NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

COURSE MATERIALS

EC 302: DIGITAL COMMUNICATION

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Provide well versed, communicative Electronics Engineers with skills in Communication systems with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1. To prepare students to excel in postgraduate programmes or to succeed in industry / technical profession through global, rigorous education and prepare the students to practice and innovate recent fields in the specified program/ industry environment.

PEO2. To provide students with a solid foundation in mathematical, Scientific and engineering fundamentals required to solve engineering problems and to have strong practical knowledge required to design and test the system.

PEO3. To train students with good scientific and engineering breadth so as to comprehend, analyze, design, and create novel products and solutions for the real life problems.

PEO4. To provide student with an academic environment aware of excellence, effective communication skills, leadership, multidisciplinary approach, written ethical codes and the life-long learning needed for a successful professional career.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Facility to apply the concepts of Electronics, Communications, Signal processing, VLSI, Control systems etc., in the design and implementation of engineering systems.

PSO2: Facility to solve complex Electronics and communication Engineering problems, using latest hardware and software tools, either independently or in team.optimization.

COURSE OUTCOMES
EC 302

SUBJECT CODE: EC 302	
COURSE OUTCOMES	
C302.1	Ability to illustrate the Digital representation of analog source and compare the performance of various Digital Pulse Modulation Schemes.
C302.2	Ability to quantify Inter Symbol Interference (ISI) problem in digital communication and to derive the Nyquist Criteria for zero ISI in data transmission and to apply ISI in controlled manner.
C302.3	Ability to illustrate signal space representation of signal using Gram Schmidt orthonormalisation procedure.
C302.4	Ability to analyse the error probability for different modulation schemes like BPSK, BFSK, QPSK and so on.
C302.5	Ability to the apply principle of spread spectrum communication and to illustrate the concepts of FHSS and DSSS.
C302.6	Ability to analyze fading channel and associated time varying characteristics and reception of signals and investigate various Multiple Access Techniques.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO’S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C302.1			3									1
C302.2	2	3	3		2			2				1
C302.3	2	3	3	3	2	3	2				2	1
C302.4		3	3	3		3						1
C302.5			3		2							1
C302.6			3	3			2					1
C302	2	3	3	3	2	3	2	2			2	1

CO’S	PSO1	PSO2	PSO3
C302.1			
C302.2	3		
C302.3	3	3	2
C302.4	3	3	2
C302.5			
C302.6			
C302	3	3	2

SYLLABUS

COURSE CODE	COURSE NAME	L-T-P-C	YEAR OF INTRODUCTION
EC302	**Digital Communication**	**4-0-0-4**	**2016**

Prerequisite: EC204 Signals and Systems, EC208 Analog Communication

Course Objectives:

- To understand the concept of Digital representation of analog source
- To understand the Performance comparison various pulse modulation schemes
- To discuss Inter Symbol Interference (ISI) problem in digital communication and to derive the Nyquist Criteria for zero ISI in data Transmission
- To analyse the need for introducing ISI in controlled manner
- To understand signal space representation of signal using Gram Schmidt orthonormalisation procedure
- To analyse the error probability for different modulation schemes like BPSK, BFSK, QPSK etc.
- To understand the principle of spread spectrum communication and to illustrate the concept of FHSS and DSSS
- To understand various Multiple Access Techniques

Syllabus: Overview of Random variables and Random process, Overall picture and relevance of digital communication, Digital Pulse modulation, Signal space concepts, Matched filter receiver, Review of Gaussian random process, Digital band pass modulation schemes, Detection of signals in Gaussian noise, Pseudo–noise sequences, Importance of synchronization, Spread spectrum communication, Diversity techniques, Multiple Access Techniques.

Expected Outcome

The students will be able to

i. Illustrate the Digital representation of analog source
ii. Compare the performance of various Digital Pulse Modulation Schemes
iii. Apply the knowledge of ISI problems in Digital communication to derive Nyquist criteria for zero ISI
iv. Analyze the need for introducing ISI in Digital Communication in a controlled manner
v. Construct signal space representation of signal using Gram Schmidt orthonormalisation procedure
vi. Compare the error probability for different digital modulation schemes like BPSK, BFSK, QPSK etc.
vii. Describe the principle of spread spectrum communication and to illustrate the concept of FHSS and DSSS
viii. Understand various Diversity Techniques

Text Books:

1. John G. Proakis, Masoud Salehi, Digital Communication, McGraw Hill Education Edition, 2014
2. Nishanth N, Digital Communication, Cengage Learning India , 2017
3. Ramakrishna Rao, Digital communication, Tata McGraw Hill Education Pvt. Limited.
4. Simon Haykin, Communication Systems, 4/e Wiley India, 2012.

References:

1. Couch: Analog and Digital Communication. 8e, Pearson Education India, 2013.
2. H.Taub and Schilling Principles of Communication Systems, , TMH, 2007
3. K.Sam Shanmugham, Digital and Analog Communication Systems, John Wiley & Sons
4. Pierre Lafrance, Fundamental Concepts in Communication, Prentice Hall India.
5. Sheldon.M.Ross, "Introduction to Probability Models", Academic Press, 7th edition.
6. Sklar: Digital Communication, 2E, Pearson Education.
7. T L Singal, Digital Communication, McGraw Hill Education (India) Pvt Ltd, 2015

Course Plan

Module	Course content	Hours	End Sem. Exam Marks
I	**Overview of Random variables and Random process:** Random variables–continuous and Discrete, random process-Stationarity, Autocorrelation and power spectral density, Transmission of Random Process through LTI systems, PSD, AWGN	3	15
	Pulse Code Modulation (PCM): Pulse Modulation, Sampling process, Performance comparison of various sampling techniques Aliasing, Reconstruction, PAM, Quantization, Noise in PCM system	3	
	Modifications of PCM: Delta modulation, DPCM, ADPCM, ADM, Performance comparison of various pulse modulation schemes, Line codes, PSD of various Line codes	4	
II	**Transmission over baseband channel:** Matched filter, Inter Symbol Interference (ISI), Nyquist Criteria for zero ISI, Ideal solution, Raised cosine spectrum, Eye Pattern	4	15
	Correlative Level Coding - Duobinary coding, precoding, Modified duobinary coding, Generalized Partial response signalling.	3	
	FIRST INTERNAL EXAM		
III	**Signal Space Analysis:** Geometric representation of signals, Gram Schmidt orthogonization procedure.	3	15
	Transmission Over AWGN Channel: Conversion of the continuous AWGN channel into a vector channel, Likelihood function, Maximum Likelihood Decoding, Correlation Receiver	4	
IV	**Digital Modulation Schemes:** Pass band transmission model**,** Coherent Modulation Schemes- BPSK, QPSK, BFSK**.** Non-Coherent orthogonal modulation schemes**,** Differential Phase Shift Keying (DPSK)	4	15
	Detection of Binary modulation schemes in the presence of noise**,** BER for BPSK, QPSK, BFSK	5	
	SECOND INTERNAL EXAM		
V	**Pseudo–noise sequences**: Properties of PN sequences. Generation of PN Sequences, generator polynomials, Maximal length codes and Gold Codes.	3	20

	Importance of synchronization: Carrier, frame and symbol/chip synchronization techniques.	2	
	Spread spectrum communication: Direct sequence spread spectrum with coherent binary phase shift keying, Processing gain, Probability of error, Anti-jam Characteristics, Frequency Hop spread spectrum with MFSK, Slow and Fast frequency hopping.	4	
VI	**Multipath channels:** classification, Coherence time, Coherence bandwidth, Statistical characterization of multi path channels, Binary signalling over a Rayleigh fading channel.	3	**20**
	Diversity techniques: Diversity in time, frequency and space.	2	
	Multiple Access Techniques: TDMA, FDMA, CDMA and SDMA – RAKE receiver, Introduction to Multicarrier communication- OFDM	5	
END SEMESTER EXAM			

Question Paper Pattern (End Semester Exam) Maximum

Marks : 100 **Time : 3 hours**

The question paper shall consist of three parts. Part A covers modules I and II, Part B covers modules III and IV, and Part C covers modules V and VI. Each part has three questions uniformly covering the two modules and each question can have maximum four subdivisions. In each part, any two questions are to be answered. Mark patterns are as per the syllabus with 30% for theory and 70% for logical/numerical problems, derivation and proof.

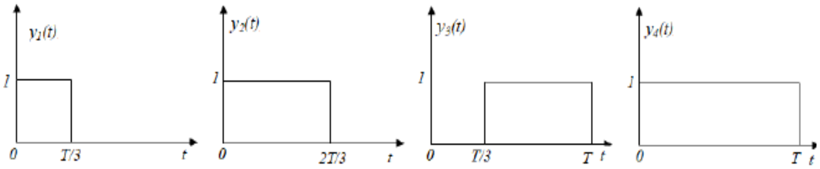
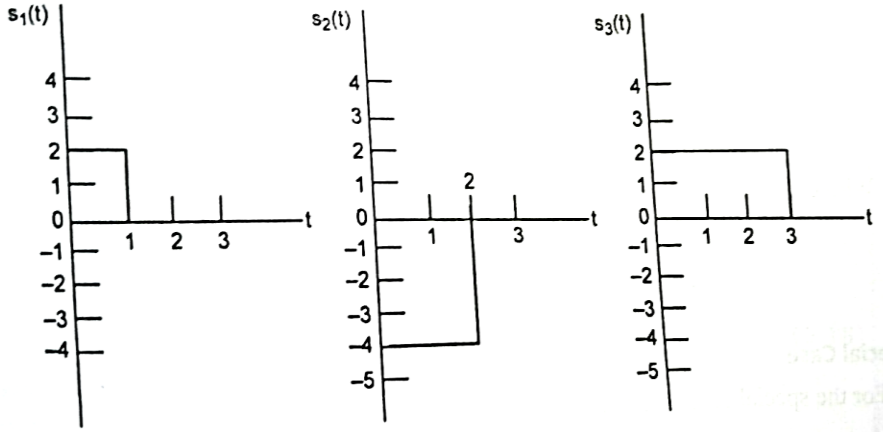
QUESTION BANK

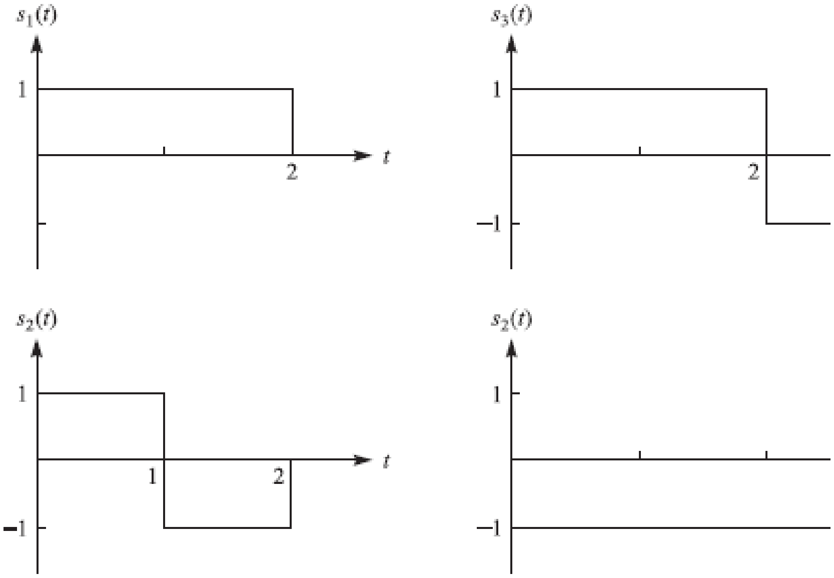
MODULE I				
Q:NO:	QUESTIONS	CO	KL	PAGE NO:
1	A strictly band limited voice signal with highest frequency component of 3.1 KHz is sampled at the standard sampling rate of Nyquist rate plus 1.8 kHz, to avoid Aliasing and then non-uniformly quantized. The quantized samples are then encoded using 8-bits per sample. Estimate the data rate for the PCM transmission.	CO1	K3	50
2	Explain Random variables and provide the relationship between Sample space, Random variable and Probability.	CO1	K2	2
3	Explain Aliasing error in the Sampling process with diagrams and the way to tackle it	CO1	K2	58
4	State and explain the Sampling techniques used with the aid of diagrams and appropriate equations.	CO1	K1	47
5	Define Variance of a Random variable and obtain the relationship between Variance, Mean and Mean-Square value.	CO1	K1	11
6	State the need for Line Codes and with the help of example 01101001, explain and display the Unipolar NRZ Signalling, Polar NRZ Signalling and Unipolar RZ Signalling	CO1	K2	11
7	State and Prove Shannon's Sampling Theorem	CO1	K2	54
8	Describe the Pulse Code Modulation with the aid of a block diagram	CO1	K2	7
9	With the aid of a diagram explain the concept of Random process	CO1	K2	14
10	State Einstein Wiener Khintchine laws and apply these laws to establish Power Spectral Density and hence prove its properties.	CO1	K3	28

11	Define Pulse Amplitude Modulation (PAM) and analyze the operations of sampling and reconstruction with the help of diagrams and related equations.	CO1	K4	3
12	State the need for Line Codes and with the help of example 01101001, explain and display the Bipolar RZ signalling, Split-phase or Manchester code	CO1	K2	27
13	Define Autocorrelation function. Explain its properties	CO1	K2	19
14	Explain Power Spectral Density of a Random Process.	CO1	K2	26
15	Explain Line Codes used in Digital Communication with the aid of an example.	CO1	K4	
16	Given two Random processes X(t) and θ that are independent, another Random process is obtained by mixing X(t) with a sinusoidal wave with random phase as $Y(t) = X(t).Cos(2\pi f_c t + \theta)$, where fc is the linear frequency. Estimate the ACF and the PSD of Y(t) in terms of those of the Random process X(t)	CO1	K5	33

MODULE II

1	Analyze ISI and its problems in modern digital communications to the band pass signals. Use diagrams and equations to validate your answer.	CO2	K4	65
2	Explore Eye Patterns with the aid of a well labeled diagram and provide its applications for Digital Communications.	CO2	K3	87
3	Demonstrate pure Duobinary coding and decoding for the following sequence $\{x_k\}$ = 0 1 0 0 1 1 0. Let the first bit be a start-up digit, not part of the data.	CO2	K3	92
4	With necessary expressions, analyze the practical difficulties encountered in ideal Nyquist channel and how they are overcome by Raised Cosine Filters.	CO2	K4	77
5	State the need for a Precoder in a Duobinary signalling system. Analyze the Precoder based Duobinary signalling systems with the aid of diagram and related equations.	CO2	K4	89
6	For the given input binary data 1011101, obtain the output of the modified Duobinary encoder using Precoding. Scrutinize how the data can be detected at the receiver at the decoder.	CO2	K3	95
7	Analyze the Generalized Partial response signaling with the aid of diagrams and related equations.	CO2	K4	97
8	Analyze Duobinary Signaling with the aid of a block diagram and related equations	CO2	K4	41
9	Evaluate Pulse Shaping to reduce ISI	CO2	K6	78

10	Analyze the Matched Filter with the aid of appropriate diagrams and equations.	CO2	K4	67
11	Perform the analysis of Nyquist Criteria for zero ISI with the aid of appropriate diagrams and equations.	CO2	K4	76
12	Compare and contrast Duobinary Coding and Binary Signaling.	CO2	K6	99
	MODULE III			
1	Analyze Geometric Representation of Signals with the aid of diagrams and appropriate equations.	CO3	K4	104
2	Consider the signals $y_1(t),y_2(t),y_3(t)\&y_4(t)$ given below.  Find the orthonormal basis for these set of signals using Gram-Sc orthogonization procedure.	CO3	K5	108
3	Analyze in detail the Gram-Schmidt Orthogonalization procedure.	CO3	K4	111
4	***(a)*** ***Using the Gram-Schmidt orthogonalization procedure, find a set of orthonormal basis functions to represent the three signals $s_1(t)$, $s_2(t)$, and $s_3(t)$ shown in figure 6.5.*** ***(b)*** ***Express each of these signals in terms of the set of basis functions found in part (a).***  	CO3	K5	114
5	Explain the reasons for the steady transition from Analog Communications to Digital Communications	CO3	K2	63

6	Describe with the aid of suitable diagrams and equations, the schemes of Analyzer and Synthesizer	CO3	K2	105
7	Analyze the Conversion of the Continuous AWGN Channel into a Vector Channel.	CO3	K4	118
8	Draw the Correlator structure at the Receiver detector and derive the relations for Mean and Variance of the signal at the Correlator j output.	CO3	K3	132
9	Apply the Gram-Schmidt procedure to the set of four wavefor below.  	CO3	K4	115
10	State The Schwarz Inequality and prove it from first principles for real-valued signals.	CO3	K3	116

MODULE IV

1	What is baseband transmission?	CO4	K1	136
2	What is passband transmission?	CO4	K1	136
3	State the need for Modulation and the types of Keying.	CO4	K1	136
4	Explain the need for Mapping and the relevance of a Memoryless	CO4	K2	137

	channel.			
5	Define ASK, PSK and FSK and illustrate with the aid of diagrams	CO4	K2	138
6	State the differences between coherent and non-coherent reception of signals.	CO4	K1	139
7	With the aid of a block diagram, describe the passband model for digital communication with the use of relevant equations.	CO4	K2	140
8	Analyze the BPSK digital modulation with the aid of constellation diagram, transmitter and receiver.	CO4	K4	141
9	Derive the BER for BPSK modulation from first principles with the help of state space diagram for coherent reception.	CO4	K3	144
10	Analyze the BFSK digital modulation with the aid of constellation diagram, transmitter and receiver.	CO4	K4	149
11	Derive the BER for BFSK modulation from first principles with the help of state space diagram for coherent reception.	CO4	K3	154
12	Analyze the QFSK digital modulation with the aid of constellation diagram, transmitter and receiver.	CO4	K4	165
MODULE V				
1	State the need for Spread Spectrum Communications.	CO5	K1	197
2	Explain the advantage of SS communication in its ability to reject interference when compared to plain old digital modulation.	CO5	K2	198
3	State the need for PN Sequence and provide the diagram of its generation with explanation.	CO5	K2	199
4	Explain the maximal-length-sequence.	CO5	K2	200

5	Analyze the properties of Maximal-Length Sequences	CO5	K4	203
6	Explain about choosing a Maximal-Length Sequence	CO5	K2	207
7	With the aid of Gold's Theorem, explain the Gold Codes.	CO5	K2	212
8	Explain the need for Synchronization in Digital Communications and the types.	CO5	K2	217
9	Describe the Carrier Synchronization with the aid of block diagrams and supporting analysis.	CO5	K4	218
10	Describe the Symbol Synchronization with the aid of block diagrams and supporting analysis.	CO5	K4	222
	MODULE VI			
1	Explain about time varying channels.	CO6	K2	245
2	Describe the characterization of fading multipath channels with the aid of supporting analyses.	CO6	K4	246
3	State the types of mathematical models adopted for fading channels.	CO6	K1	249
4	Analyze in detail the Channel Correlation Functions and Power Spectra with the aid of supporting mathematics.	CO6	K4	250
5	Explain the differences between slow fading channel and fast fading channels.	CO6	K2	255
6	Analyze the Statistical Models for Fading Channels.	CO6	K4	258
7	Performance comparison of binary signalling on a Rayleigh fading channel	CO6	K5	274
9	Explain Diversity and Diversity techniques used for fading channels.	CO6	K3	280
10	Compare and contrast the various multiple access techniques such as FDMA, SDMA, TDMA and CDMA.	CO6	K6	281

APPENDIX 1

CONTENT BEYOND THE SYLLABUS

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Overview of Random variables and Random process:

Module-I Part-1

EC302 Digital Communication

Review of Probability Theory

- Communication systems deal with quantities that are not deterministic and hence there is a need to apply non-deterministic (random) and probabilistic approaches.
- You should review the basics of Probability that you have already learnt, be aware that probability of an event always lies between 0 and 1.
- Also you should be aware of independent events, statistically uncorrelated quantities.
- Awareness of Conditional Probability and Bayes theorem.
- Awareness of Basic rules and formulations for the Probability of events.

Random Variables

- When a random experiment is performed, there are various outcomes possible.
- It is convenient to consider the expt and its possible outcomes as defining a space and its points.
- With the kth outcome of the expt, there is a point associated called <u>Sample Point</u>, denoted by s_k.
- The totality of sample points corresponding to the aggregate of all possible outcomes of the expt is called <u>Sample Space</u> (S).
- An <u>Event</u> corresponds to either a single sample point or a set of sample points.
- The outcome of an expt can be a variable that can wander over a set of sample points and whose value is determined by the expt.

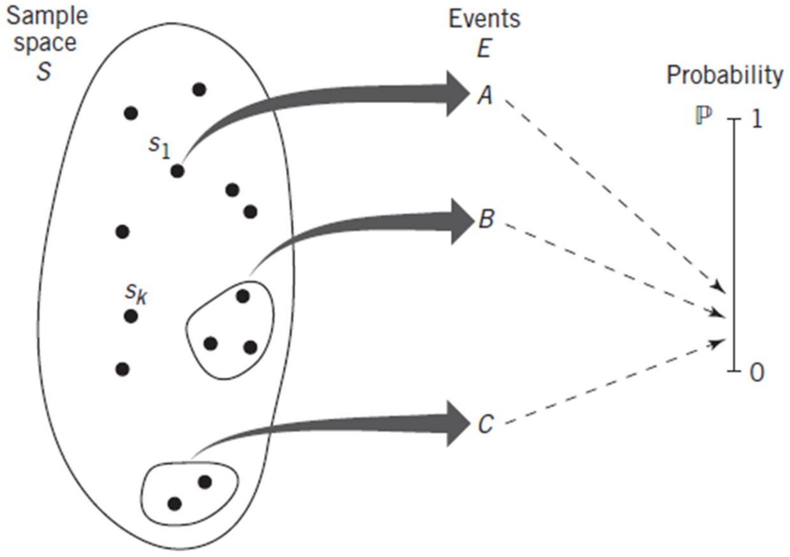


Illustration of the relationship between sample space, events, and probability

Random Variable

- A function whose domain is a sample space and whose range is some set of real numbers is called a random variable of the expt.
- When the outcome of an experiment is ‘s’, the random variable is denoted by X(s) or X.
- The random variable (rv) can be:
  - a discrete random variable (or)
  - a continuous random variable.

Discrete Random Variable

- Consider, for example, the sample space that represents the integers 1, 2, …, 6, each one of which is the number of dots that shows uppermost when a die is thrown.
- Let the sample point s_k denote the event that *k* dots show in one throw of the die.
- The random variable used to describe the probabilistic event s_k in this experiment is said to be a Discrete Random Variable.

Continuous Random Variable

- Consider the electrical noise being observed at the front end of a communication receiver.
- The random variable, representing the amplitude of the noise voltage at a particular instant of time, occupies a continuous range of values, both positive and negative.
- The random variable representing the noise amplitude is said to be a Continuous Random Variable.

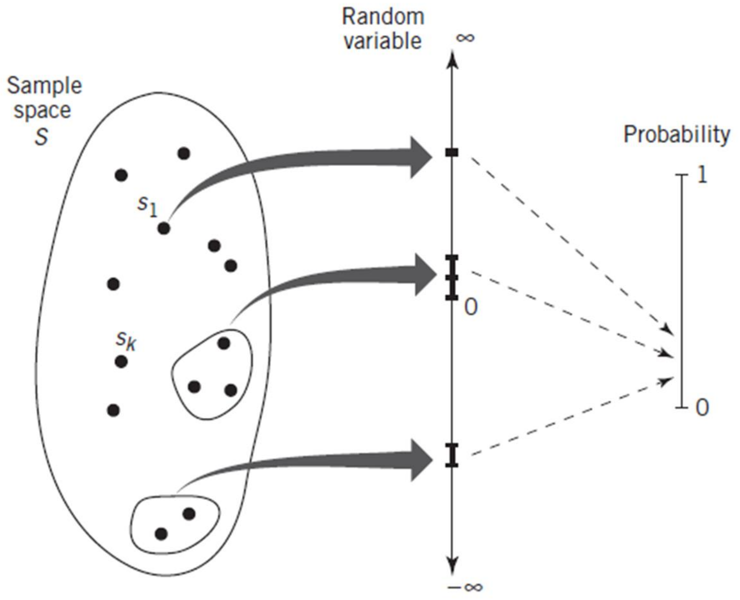


Illustration of the relationship between sample space, random variables, and probability.

- Consider the random variable X and the probability of the event $X \leq x$.
- Denote this probability by P[$X \leq x$].

  F_X(x)= P[$X \leq x$] for all x ………(1)
- The function $F_X(x)$ is called the Cumulative Distribution Function or the Distribution Function of the random variable X.
- Note that $F_X(x)$ is a function of x, not of the random variable X.
- For any point x in the sample space, the distribution function $F_X(x)$ expresses the probability of an event.

- The distribution function $F_X(x)$, applicable to both continuous and discrete random variables, has two fundamental properties:
- 1. $F_X(x)$ is a bounded function that lies between zero and one. $0 \leq F_X(x) \leq 1$
- 2. It is a monotone non-decreasing function of x

$$F_X(x_1) \leq F_X(x_2) \text{ if } x_1 \leq x_2$$

- The random variable *X* is said to be *continuous* if the distribution function $F_X(x)$ is differentiable with respect to *x* everywhere, as shown by,

$$f_X(x) = \frac{d}{dx}F_X(x) \qquad \text{for all } x \qquad \text{.........(2)}$$

- The new function $f_X(x)$ is called the <u>Probability Density Function</u> (pdf) of the random variable *X*.
- The probability of the event $x_1 < X \leq x_2$ is

$$P(x_1 < X \leq x_2) = P(X \leq x_2) - P(X \leq x_1)$$

$$= F_X(x_2) - F_X(x_1)$$

$$= \int_{x_1}^{x_2} f_X(x).dx \qquad \text{.........(3)}$$

$$\therefore F_X(x) = \int_{-\infty}^{x_2} f_X(x).dx$$

Also, $F_X(-\infty) = 0$, $F_X(\infty) = 1 \Longrightarrow \int_{-\infty}^{\infty} f_X(x).dx = 1$

The probability density function pdf must always be a <u>non-negative function</u> with a <u>total area of unity</u>.

Multiple Random Variables

- Consider two random variables X and Y.
- The joint distribution function $F_{X,Y}(x,y)$ is the probability that the random variable X is less than or equal to a specified value x, and that the random variable Y is less than or equal to another specified value y.
- $F_{X,Y}(x) = P[X \leq x, Y \leq y]$ for all x and y …(4)

- The Joint Probability Density Function of the random variables X and Y is given by,

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \quad \text{……….(5)}$$

- The joint distribution function $F_{X,Y}(x,y)$ is a monotone non-decreasing function of both x and y.
- The joint probability density function $f_{X,Y}(x,y)$ is always non-negative.
- Also, the total volume under the graph of a joint probability density function must be unity.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1 \quad \text{……….(6)}$$

- The *marginal* probability density functions are $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,\,Y}(x, y)\,\mathrm{d}y \quad \text{……….(7)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,\,Y}(x,y)\,\mathrm{d}x \quad \text{……….(8)}$$

Conditional Probability Density Function

- Suppose that X and Y are two continuous random variables with their joint probability density function $f_{X,Y}(x, y)$.
- The *conditional probability density function* of Y, such that $X = x$, is defined by,

$$f_Y(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} \quad \text{……….(9)}$$

- Provided that $f_X(x) > 0$, where $f_X(x)$ is the marginal density of X;

$$f_Y(y|x) \geq 0 \qquad \int_{-\infty}^{\infty} f_Y(y|x)\mathrm{d}y = 1$$

- From eqn (9) obtain multiplication rule.

$$f_{X,Y}(x, y) = f_Y(y|x) f_X(x)$$

- Suppose that knowledge of the outcome of *X* can, in no way, affect the distribution of *Y*.
- Then, the conditional probability density function $f_Y(y|x)$ reduces to the marginal density $f_Y(y)$,

$$f_Y(y|x) = f_Y(y)$$

- In such a case, express the joint probability density function of the random variables *X* and *Y* as the product of their respective marginal densities; i.e.,

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Golden Rules:

1. If the joint probability density function of the random variables *X* and *Y* equals the product of their marginal densities, then *X* and *Y* are Statistically Independent.

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

2. The summation of two independent continuous random variables leads to the Convolution of their respective probability density functions.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, \mathrm{d}x$$

Statistical Average

- The *expected value* or *mean* of a continuous random variable *X* is defined by,
  μ_X=E[X]=$\int_{-\infty}^{\infty} x.f_X(x).dx$, $f_X(x)$→pdf of *x* ....(10)
- The expectation of a sum of random variables is equal to the sum of individual expectations.
- The expectation of the product of two statistically independent random variables is equal to the product of their individual expectations.

- let *X* denote a random variable and let *g*(*X*) denote a real-valued function of *X* defined on the real line.
- The quantity obtained by letting the argument of the function *g*(*X*) be a random variable is also a random variable,

Let Y=g(X)

E[Y]=$\int_{-\infty}^{\infty} y.f_Y(y).dy$, where $f_Y(y)$→pdf of *y*

Another way → E[g(X)]= $\int_{-\infty}^{\infty} g(x).f_X(x).dx$ ……(11)

- This is called Expected value rule;

Second-order Moments

- For the special case of $g(X) = X^n$, application of eqn(11) leads to the *n*th *moment* of prob distribution of a random variable *X*,

$E[X^n]= \int_{-\infty}^{\infty} x^n . f_X(x). dx$ ……(12)

- Put n=1 $\Longrightarrow$ $E[X]= \int_{-\infty}^{\infty} x . f_X(x). dx$ → Mean!
- Put n=2 $\Longrightarrow$ $E[X^2]= \int_{-\infty}^{\infty} x^2 . f_X(x). dx$ → Mean Square Value! ……(13)

Variance

- Central Moments are the moments of the difference between a random variable *X* and its mean μ_X.
- The *n*th Central Moment of *X* is,

$E[(X- \mu_X)^n]= \int_{-\infty}^{\infty} (x - \mu_X)^n . f_X(x). dx$……(14)

- The Second Central Moment or Variance is,

$Var[X]=E[(X- \mu_X)^2]= \int_{-\infty}^{\infty} (x - \mu_X)^2 . f_X(x). dx$ …..(15)

- The variance of a random variable *X* is commonly denoted by σ_X^2.
- The square root of the variance, namely σ_X , is called the *standard deviation* of the random variable *X*.

σ_X^2 =Var[X]=E[(X- μ_X)^2]
= E[X^2 - 2μ_XX + μ_X^2)
= E[X^2]- 2μ_XE[X] + μ_X^2
= E[X^2]- 2μ_X^2 + μ_X^2
= E[X^2]- μ_X^2

If the mean is zero, the variance and mean square value of the rv X are equal.

Joint Moments

- The joint moment of a pair of rvs X and Y is the expectation of X^iY^k, where *i* and *k* are positive integer values.

∴ $E[X^i Y^k]= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i\, y^k. f_{X,Y}(x,y).\, dx.dy$ ……(16)

- Correlation is defined as E[XY] corresponds to i=k=1
  →$E[XY]= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y. f_{X,Y}(x,y).\, dx.dy$ ……(17)
- Covariance of rvs X and Y is defined as the Correlation of the centered rvs X-E[X] and Y-E[Y],

Cov[XY]= E[(X-E[X]).(Y-E[Y])]= E[(X- μ_X).(Y- μ_Y)] ……(18)

∴ Cov[XY]= E[XY]-$\mu_X\mu_Y$……(19)

- Correlation Coefficient of rvs X and Y is the Covariance of X and Y, normalized w.r.t product of their Standard deviations.

$$\therefore\ \rho = \frac{\mathrm{Cov[XY]}}{\sigma_X \sigma_Y} \qquad \text{......(20)}$$

- Two rvs X and Y are Uncorrelated iff their Covariance is zero
- Two rvs X and Y are Orthogonal iff their Correlation is zero, i.e., E[XY] = 0
- If rvs X and Y are Statistically Independent, then they are Uncorrelated, but the converse is not necessarily true.
- If one of the rvs *X* and *Y* or both have zero mean, and if they are orthogonal, then they are uncorrelated, and vice versa.

Stochastic or Random Processes

- A Stochastic Process is a set of Random Variables indexed in time.
- In performing a random expt, the Sample space is considered and each outcome of the expt is associated with a Sample point.
- The totality of Sample points corresponds to the aggregate of all possible outcomes of the expt is called the Sample space.
- Each Sample point of the Sample space is a function of time. (Main difference compared to Random variable!)
- The Sample space or Ensemble composed of functions of time is called a Random or Stochastic Process.

- Consider a random expt specified by the outcomes 's' from sample space 'S'.
- Assign to each Sample point 's', a function of time,

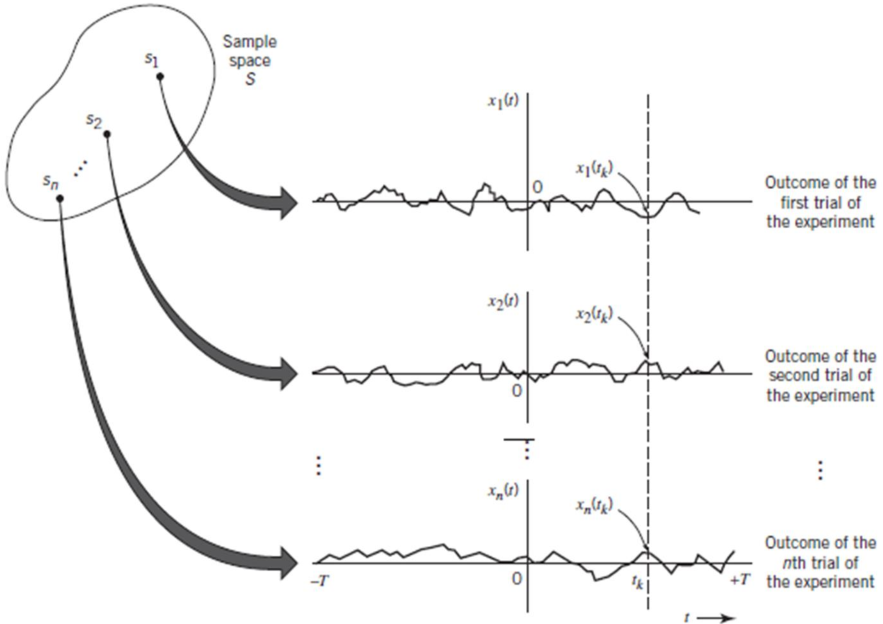
  X(t,s), $-T \leq t \leq T$,

  where 2T is the total observation interval.
- So, each Sample function is, $x_i(t)=X(t, s_i)$
- Consider a set of sample fns $\{x_i(t)|_{i=1,2,\ldots,n}\}$
- For a fixed time t_k, within the observation interval, the set of numbers,

$\{x_1(t_k), x_2(t_k),\ldots., x_n(t_k)\}=\{X(t_k, s_1), X(t_k, s_2),\ldots, X(t_k, s_n)$

→constitute a Random variable (rv).

- The Indexed Ensemble of rv, {X(t, s)} is called Random Process.
- The Random Process is simplified as X(t)



An Ensemble of Sample Functions

- The <u>Random Process </u>X(t) is an <u>Ensemble of time functions</u> together with a <u>Probability rule</u> that assigns a probability to any event associated with an observation of one of the sample functions of the stochastic process.
- For a <u>Random Variable</u>, the outcome of a random expt is mapped into a number, where as for a <u>Random Process</u>, the outcome of the random expt is mapped into a waveform that is a function of time.

- In dealing with random processes in the real world, it is found that the statistical characterization of a process is independent of the time at which observation of the process is initiated.
- That is, if such a process is divided into a number of time intervals, the various sections of the process exhibit essentially the same statistical properties.
- Such a stochastic process is said to be Stationary.
- Otherwise, it is said to be Non-Stationary.

- A stationary process arises from a stable phenomenon that has evolved into a steady-state mode of behavior, whereas a non-stationary process arises from an unstable phenomenon.
- Consider a random process $X(t)$ initiated at $t=-\infty$
- Let $X(t_1)$, $X(t_2)$, …, $X(t_k)$ denote the random variables obtained by sampling the process $X(t)$ at times t_1, t_2, …, t_k, respectively.
- The joint (cumulative) distribution function of this set of rvs is $F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$.

- Shift all the sampling times by a fixed amount τ denoting the *time shift*, thereby obtaining the new set of random variables: $X(t_1+\tau)$, $X(t_2+\tau)$, …, $X(t_k+\tau)$.
- The joint distribution function of this set of rvs is $F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k)$.
- The random process $X(t)$ is said to be stationary in the strict sense, or Strictly Stationary, if the invariance condition holds.

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) \quad \ldots\ldots(21)$$

- for all values of time shift τ, all positive integers k, and any possible choice of sampling times t_1, …, t_k.

- A random process $X(t)$, initiated at time $t=-\infty$, is Strictly Stationary if the joint distribution of any set of random variables obtained by observing the process $X(t)$ is invariant with respect to the location of the origin $t = 0$.
- Two random processes $X(t)$ and $Y(t)$ are *jointly strictly stationary* if the joint finite-dimensional distributions of the two sets of random variables $X(t_1)$, …. , $X(t_k)$ and $Y(t_1')$,….,$Y(t_j')$ are invariant with respect to the origin $t = 0$, for all positive integers *k* and *j,* and all choices of the sampling times $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$.

Points to Ponder

1. For *k* = 1,

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \quad \text{for all } t \text{ and } \tau \qquad \text{......(22)}$$

The first-order distribution function of a strictly stationary random process is independent of time t.

2. For *k* = 2 and τ = –t_2,

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_1-t_2)}(x_1, x_2) \quad \text{for all } t_1 \text{ and } t_2 \qquad \text{......(23)}$$

The second-order distribution function of a strictly stationary random process depends only on the time difference between the sampling instants and not on the particular times at which the random process is sampled.

Weakly Stationary Processes

- Another important class of random processes is Weakly Stationary processes.
- A random process *X*(*t*) is said to be weakly stationary if its moments satisfy the following two conditions:

1. The Mean(μ_X) of process X(t) is constant for all time 't'.

2. The Autocorrelation function of the process X(t) depends solely on the difference between any two times at which the process is sampled. $R_X(t_1,t_2)=R_X(t_2-t_1)$

- Such processes are also referred to as Wide-Sense Stationary processes in the literature.
- Such a process may not be Stationary in the strict sense.
- All Stationary processes are wide sense stationary, but every wide-sense Stationary process may not be strictly Stationary.
- Truly Stationary process cannot occur in real life.

Mean of Stationary Process

- Consider a stationary random process *X*(*t*).
- The <u>Mean</u> of the process *is* the Expectation of the random variable obtained by sampling the process at some time *t*,

$\mu_X(t)=E[X(t)]= \int_{-\infty}^{\infty} x f_{X(t)}(x).\,\mathrm{dx}$, $f_{X(t)}(x)$→pdf of X(t) ……(24)

- The pdf of a Stationary Process is independent of ‘t’

$\therefore$ $\mu_X(t)=\mu_X$ for all ‘t’ ……(25)

Autocorrelation Function

- <u>Autocorrelation Function</u> of the random process *X*(*t*) is the Expectation of the product of two random variables, $X(t_1)$ and $X(t_2)$, obtained by sampling the process *X*(*t*) at times t_1 and t_2, respectively.

  $\therefore R_X(t_1,t_2)= E[X(t_1)X(t_2)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),\,X(t_2)}(x_1, x_2)\ \mathrm{d}x_1\ \mathrm{d}x_2 \quad \text{……(26)}$$

$f_{X(t_1),\,X(t_2)}(x_1, x_2)$ ←Second order pdf of random process

- For a stationary random process, $f_{X(t_1),X(t_2)}(x_1,x_2)$ depends only on the difference between observation times t_1 and t_2.
- The Autocorrelation function of a strictly stationary process depends only on the time difference $t_2 - t_1$.
- $\therefore R_X(t_1,t_2)= R_X(t_1,t_2)$ for all t_1 and t_2 ……(27)
- The Autocovariance function of a stationary random process X(t) is given by,

$$C_X(t_1, t_2)= E[(X(t_1)-\mu_X).(X(t_2)- \mu_X)]$$
$$= R_X(t_2-t_1)- \mu_X^2 \quad ……(28)$$

- The Autocovariance function of a stationary random process depends only on the time difference (t_2-t_1).

- The autocovariance function of a weakly stationary process $X(t)$ depends only on the time difference $(t_2 – t_1)$.
- This equation also shows that knowing the mean and the autocorrelation function of the process $X(t)$, can uniquely determine the autocovariance function.
- The mean and autocorrelation function are therefore sufficient to describe the first two moments of the process.

Properties of Autocorrelation Function

$R_X(\tau)= E[X(t+\tau)X(t)]$ for all 't'

τ → represents time shift i.e., $t = t_2$; $\tau = t_1 - t_2$ ……(29)

PROPERTY 1 Mean-square Value

- The mean-square value of a stationary process X(t) is obtained from $R_X(\tau)$ simply by putting $\tau = 0$

∴ $R_X(0)= E[X^2(t)]$ ……(30)

PROPERTY 2 Symmetry

- The autocorrelation function $R_X(\tau)$ of a stationary process X(t) is an even function of the time shift τ; i.e.,

i.e., $R_X(\tau)= R_X(-\tau)$ ……(31)

∴ $R_X(\tau)= E[X(t)X(t-\tau)]$

- A graph of the autocorrelation function $R_X(\tau)$ plotted versus τ is symmetric about the origin.

PROPERTY 3 Bound on Autocorrelation Function

The autocorrelation function $R_X(\tau)$ attains its maximum magnitude at $\tau = 0$; i.e.,

$|R_X(\tau)| \leq R_X(0)$ ……(32)

PROPERTY 4 Normalization

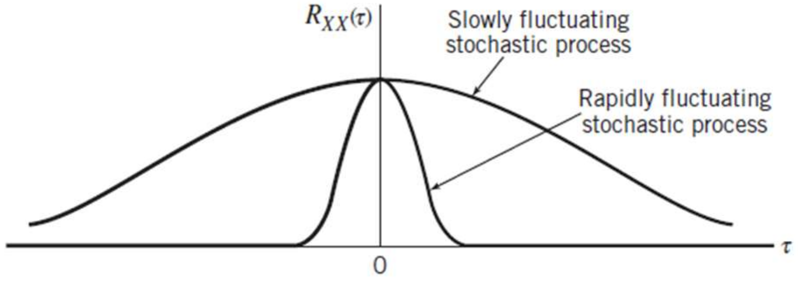
Given the normalized autocorrelation function,

$\rho_X(\tau)=\frac{R_X(\tau)}{R_X(0)}$ ……(33)

Values are confined to the range [-1, 1]

Physical Significance of the Autocorrelation Function

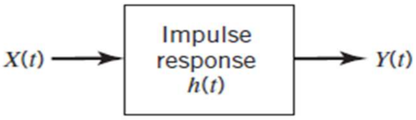
- The Autocorrelation function $R_X(\tau)$ is significant because it provides a means of describing the interdependence of two random variables obtained by sampling the stochastic process X(t) at times τ seconds apart.
- The more rapidly the random process *X*(*t*) changes with time, the more rapidly will the Autocorrelation function $R_X(\tau)$ decrease from its maximum $R_X(0)$ as τ increases.



Autocorrelation functions of slowly and rapidly fluctuating stochastic processes.

Transmission of a Random Process through a Linear Time-invariant Filter

- Let a Random process $X(t)$ be applied as input to a Linear Time-Invariant (LTI) filter of impulse response $h(t)$, to produce a new random process $Y(t)$ at the filter output.



Transmission of a random process thro' a LTI filter.

- In general, it is difficult to describe the probability distribution of the output random process $Y(t)$, even when the probability distribution of the input random process $X(t)$ is completely specified.

- Let random process X(t) be a Stationary process.
- The transmission of a process thro' a LTI filter is governed by Convolution integral, i.e., express output $Y(t)$ as,

$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\, d\tau_1$$ $\tau_1 \rightarrow$ is a local time.

- Hence, the Mean of $Y(t)$ is,

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\, d\tau_1\right] \quad \text{......(34)}$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\mathbb{E}[X(t-\tau_1)]\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\mu_X(t-\tau_1)\, d\tau_1 \quad \text{......(35)}$$

- When i/p *X*(*t*) is stationary, the mean is a constant,

∴ $\mu_X(t-\tau_1) = \mu_X(t) = \mu_X$ = constant

∴ $\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\,\mathrm{d}\tau_1 = \mu_X H(0)$ ……(36)

- where *H*(0) is the zero-frequency (dc) response of the system.

- The mean of random process *Y*(*t*) produced at the output of a LTI filter in response to a stationary process *X*(*t*), acting as the input process, is equal to the mean of *X*(*t*) times the zero-frequency(dc) response of the filter.

- Autocorrelation Function of Y(t):

$R_Y(t, u) = E[Y(t)Y(u)]$

- where *t* and *u* denote two values of the time at which the output process *Y*(*t*) is sampled.

$R_Y(t, u) =$

$$\mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,\mathrm{d}\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)\,\mathrm{d}\tau_2\right] \quad \text{……(37)}$$

$$= \int_{-\infty}^{\infty}\left[h(\tau_1)\int_{-\infty}^{\infty}\mathrm{d}\tau_2\, h(\tau_2)\mathbb{E}[X(t-\tau_1)X(u-\tau_2)]\right]\mathrm{d}\tau_1$$

$$= \int_{-\infty}^{\infty}\left[h(\tau_1)\int_{-\infty}^{\infty}\mathrm{d}\tau_2\, h(\tau_2)\, R_X\,(t-\tau_1, u-\tau_2)\right]\mathrm{d}\tau_1 \quad \text{……(38)}$$

- When the input *X*(*t*) is a stationary process, the autocorrelation function of *X*(*t*) is only a function of the difference between the sampling times $t - \tau_1$ and $u - \tau_1$.
- Put $\tau = u - t$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau + \tau_1 - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \quad ......(39)$$

- which depends only on the time difference τ

From eqn (36) and eqn(39),

- If the input to a stable LTI filter is a stationary random process, then the output of the filter is also a stationary random process.
- Use Property 1 of the Autocorrelation function $R_Y(\tau)$, it follows, that the Mean-Square value of the output process *Y*(*t*) is obtained by putting $\tau = 0$ in eqn (38),

$$R_Y(0) = \mathbb{E}[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_1 - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

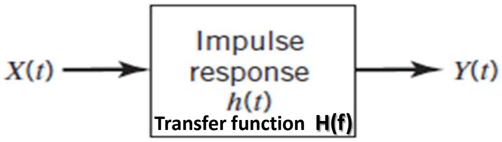
- This is a constant! ......(40)

Power Spectral Density (PSD)

- Power Spectral Density (PSD) of a Wide- Sense Stationary (WSS) Random Process X(t) is the <u>Fourier Transform</u> of the Autocorrelation Function (ACF).
- $S_X(f)$= Fourier Transform of $R_X(\tau)$

$$= \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau} d\tau \qquad ……(41)$$

LTI Filter



Transmission of a Random process thro' a LTI filter.

The impulse response of the Filter is the <u>inverse Fourier Transform</u> of Transfer function.

$$\therefore \ h(\tau_1) = \int_{-\infty}^{\infty} H(f).\, e^{j2\pi f \ \ {}_1} df \qquad ……(42)$$

From eqn(39),

$$E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f).\, e^{j2\pi f\tau_1} df \right] h(\tau_2)\, R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

$$= \int_{-\infty}^{\infty} df.\, H(f) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2) \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1)\, e^{j2\pi f\tau_1} d\tau_1$$

Let $\tau = \tau_2 - \tau_1$

∴ $E[Y^2(t)] = \int_{-\infty}^{\infty} df\, H(f) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2)\, e^{+j2\pi f\tau_2} \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau} d\tau$

$\int_{-\infty}^{\infty} h(\tau_2)\, e^{+j2\pi f\tau_2} d\tau_2 = H^*(f)$ ← Conjugate of H(f)

∴ $E[Y^2(t)] = \int_{-\infty}^{\infty} df\, |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau} d\tau$
$|H(f)|^2$ ← Square magnitude response of H(f)

$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau} d\tau$ ← Power Spectral Density!
∴ $E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$ ......(43)

Relevance of Eqn(43)

- The Mean Square value of the output of the LTI filter, in response to a Stationary Random process at its input, is equal to the integral over all frequencies of the Power Spectral Density of the input Stationary Random process multiplied by the Squared Magnitude Response of the Filter.

Einstein-Wiener-Khintchine Relations

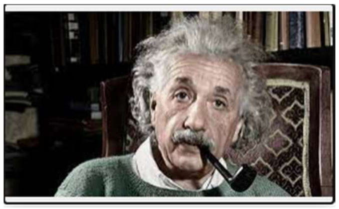


Plate-1

Plate-2

Plate-3

Norbert Wiener proved this theorem for the case of a deterministic function in 1930.

Aleksandr Khinchin later formulated an analogous result for stationary stochastic processes and published that probabilistic analogue in 1934.

Albert Einstein explained, without proofs, the idea in a brief two-page memo in 1914.

Einstein-Wiener-Khintchine Relations

- Einstein-Wiener-Khintchine (EWK) Relations gives the relationship between Power Spectral Density and Autocorrelation Function of a Stationary Random Process.
- The Power Spectral Density $S_X(f)$ and the Autocorrelation function $R_X(\tau)$ of a Stationary Random Process X(t) form a Fourier transform pair.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau} d\tau \qquad \text{......(44a)}$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{-j2\pi f\tau} df \qquad \text{......(44b)}$$

Properties of the PSD

Property 1:

The Power Spectral Density (PSD) of a Stationary random process for zero frequency (dc) value is equal to the total area under the graph of Autocorrelation function (ACF).

$$\therefore \quad S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau \quad \text{......(45)}$$

Property 2:

The Mean Square value of a Stationary random process is equal to the total area under the graph of Power Spectral Density (PSD).

$$\therefore \quad R_X(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df \quad \text{......(46)}$$

Property 3:

The Power Spectral Density (PSD) of a Stationary random process is always non-negative, i.e.,

$$S_X(f) \geq 0 \text{ for all } 'f' \quad \text{......(47)}$$

Property 4:

The Power Spectral Density (PSD) of a real-valued random process is an even function of frequency, i.e.,

$$S_X(-f) = S_X(f) \quad \text{......(48)}$$

Property 5:

The normalized Power Spectral Density (PSD) of a random process has properties associated with a probability density function.

$$p_X(\mathrm{f}) = \frac{\mathrm{S_X(f)}}{\int_{-\infty}^{\infty} \mathrm{S_X(f).df}} \geq 0, \text{ for all 'f'} \quad \text{......(49)}$$

- The total area under the function p_X(f) is unity.

PSD of Output Random Process

- S_Y(f) is the Power Spectral Density (PSD) of the o/p random process obtained by passing random process X(t) thro' an LTI filter of transfer function H(f).
- PSD of random process is equal to Fourier transform of its Autocorrelation function(ACF).

$$\therefore\ \mathrm{S_Y(f)} = \int_{-\infty}^{\infty} \mathrm{R_Y}(\tau)\, \mathrm{e}^{\mathrm{j}2\pi \mathrm{f}\tau} \mathrm{d}\tau \quad \text{......(50)}$$

Use eqn(39)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_{X.}\ (\tau + \tau_1 - \tau_2)\ \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

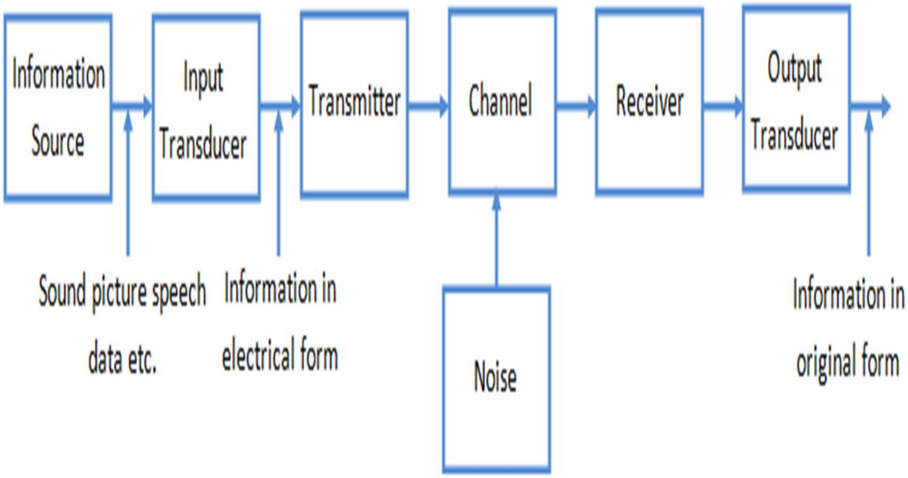
Let $\tau + \tau_1 - \tau_2 = \tau_0 \rightarrow \tau = \tau_0 - \tau_1 + \tau_2$

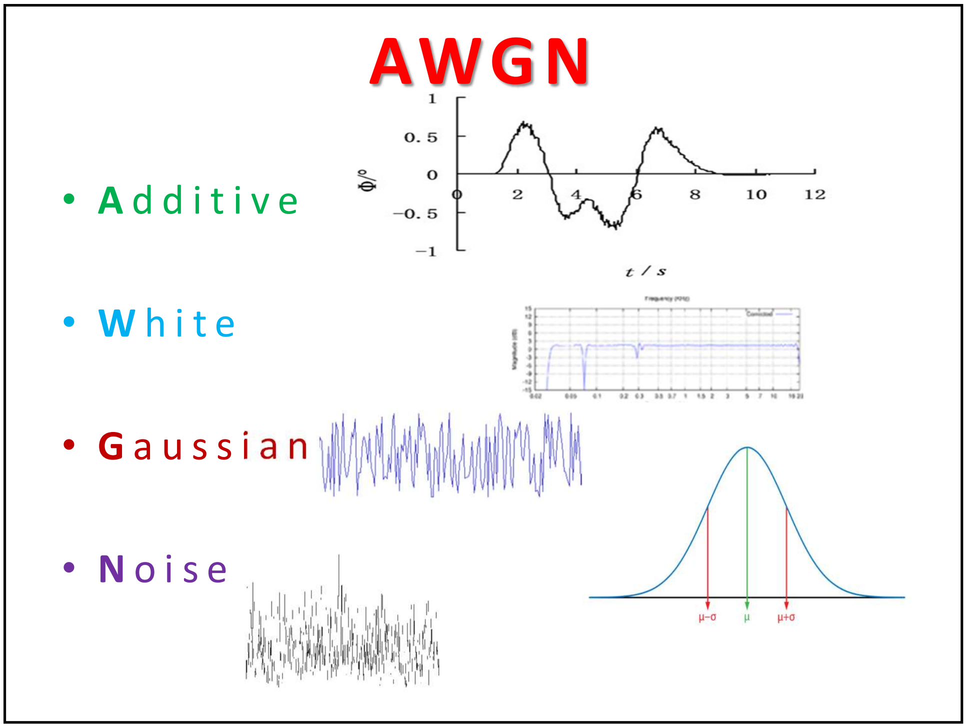
$S_Y(f) = H(f).\ H^*(f).\ S_X(f)$

$= |H(f)|^2.\ S_X(f)$ ……(51)

The Power Spectral Density of the o/p random process Y(t) is equal to the Power Spectral Density of the i/p random process X(t) multiplied with the squared magnitude response of the Filter.

Block Diagram of a Generic Digital Communication System

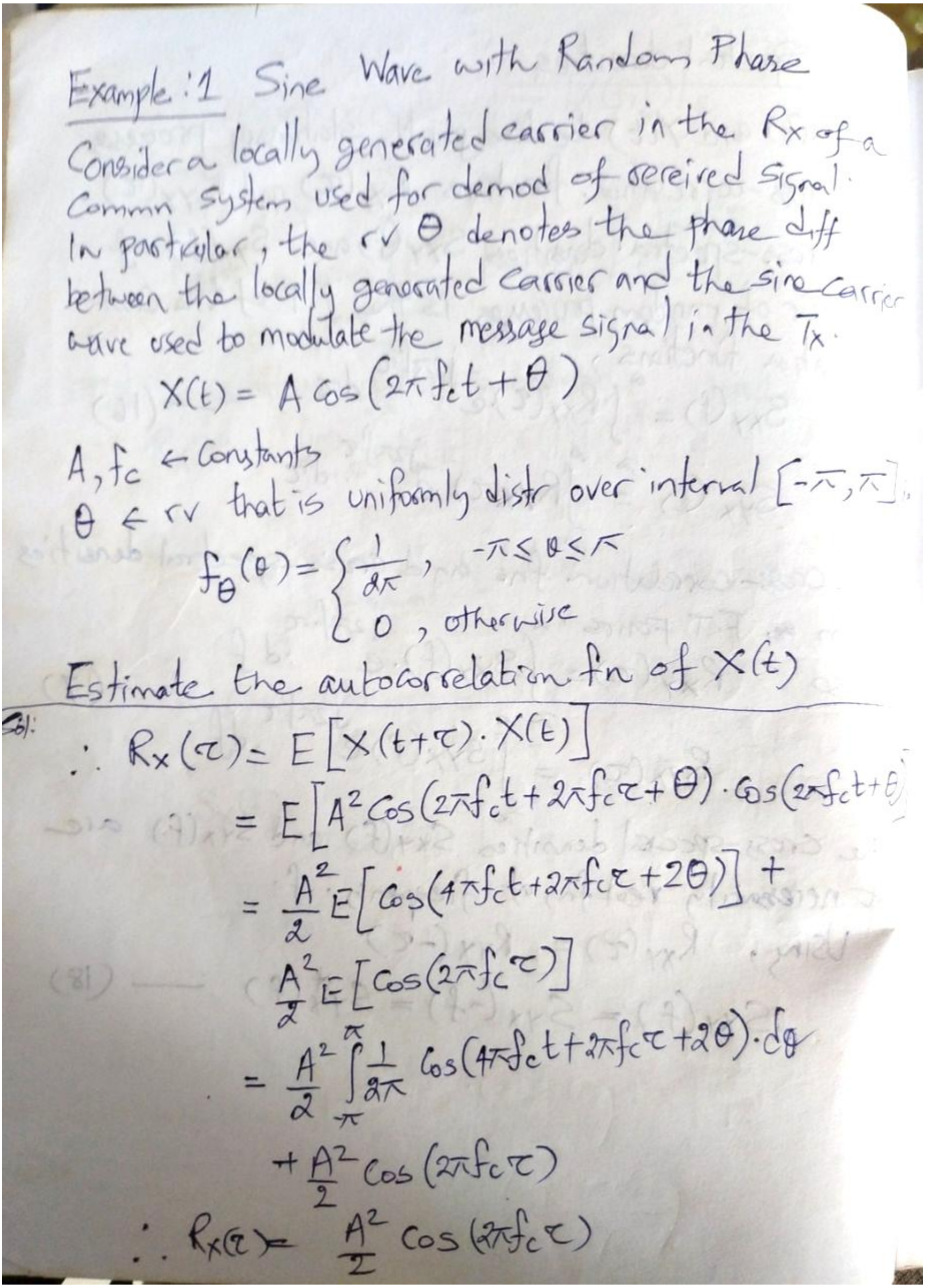




AWGN

- Additive
- White
- Gaussian
- Noise

WORKED EXAMPLES IN EC302 MODULE-1 PART-1



Example : 1 Sine Wave with Random Phase

Consider a locally generated carrier in the Rx of a Commn system used for demod of received signal. In particular, the rv θ denotes the phase diff between the locally generated carrier and the sine carrier wave used to modulate the message signal in the Tx.

$$X(t) = A\cos(2\pi f_c t + \theta)$$

A, f_c ← Constants

θ ← rv that is uniformly distr over interval $[-\pi, \pi]$.

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

Estimate the autocorrelation fn of $X(t)$

Sol:

$$\therefore R_X(\tau) = E[X(t+\tau) \cdot X(t)]$$

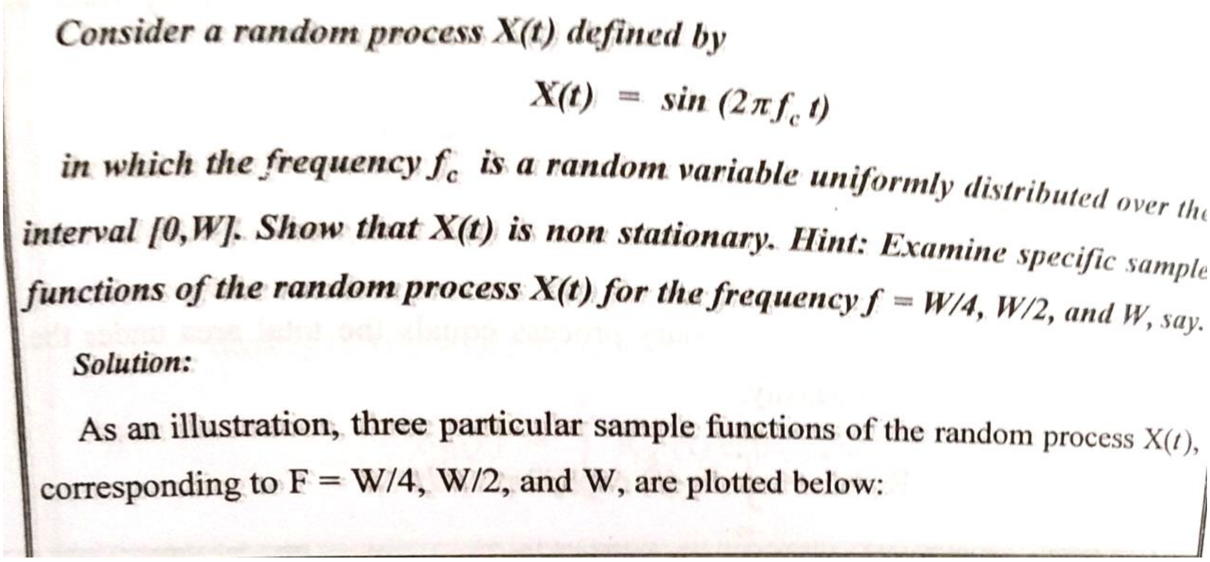
$$= E[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cdot \cos(2\pi f_c t + \theta)]$$

$$= \frac{A^2}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)] + \frac{A^2}{2} E[\cos(2\pi f_c \tau)]$$

$$= \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) \cdot d\theta + \frac{A^2}{2} \cos(2\pi f_c \tau)$$

$$\therefore R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

Example -2



Consider a random process X(t) defined by

$$X(t) = \sin(2\pi f_c t)$$

in which the frequency f_c is a random variable uniformly distributed over the interval [0,W]. Show that X(t) is non stationary. Hint: Examine specific sample functions of the random process X(t) for the frequency f = W/4, W/2, and W, say.

Solution:

As an illustration, three particular sample functions of the random process $X(t)$, corresponding to F = W/4, W/2, and W, are plotted below:

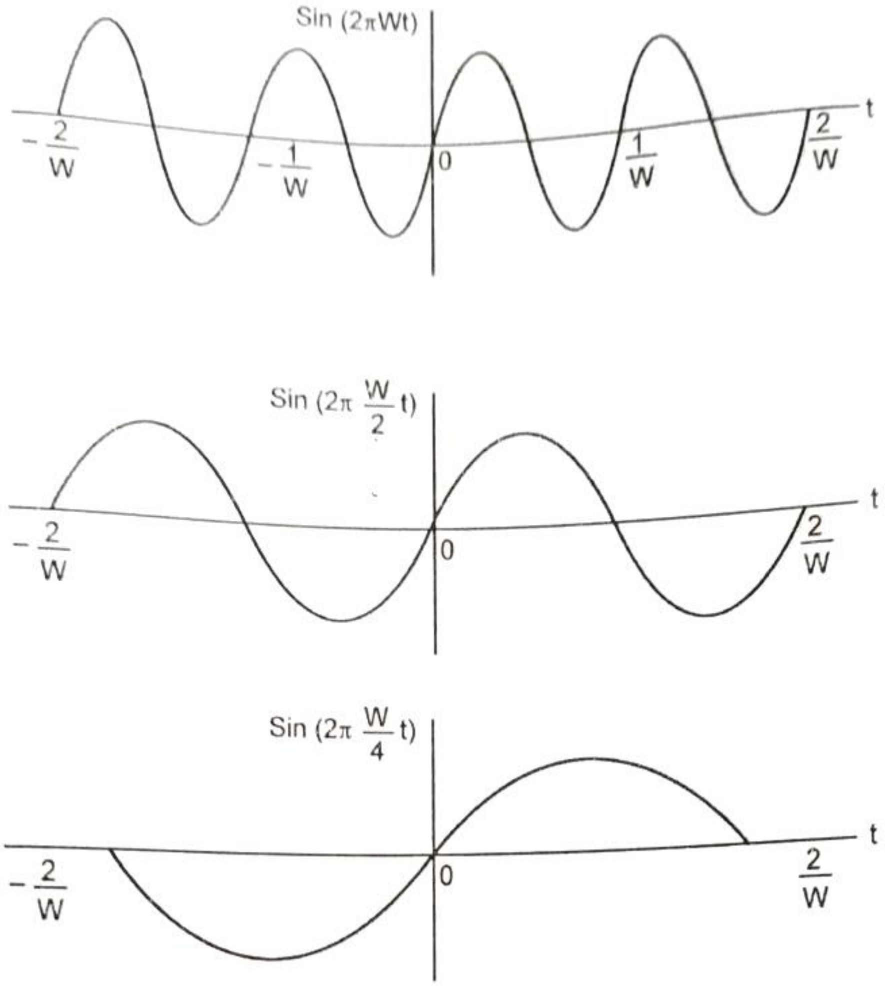


Fig. 2.6.

To show that $X(t)$ is nonstationary, we need only observe that every waveform illustrated above is zero at $t = 0$, positive for $0 < t < 1/2W$, and negative for $-1/2\ W < t < 0$. Thus, the probability density function of the random variable $X(t_1)$ obtained by sampling $X(t)$ at $t_1 = 1/4W$ is identically zero for negative argument, whereas the probability density function by the random variable $X(t_2)$ obtained by sampling $X(t)$ at $t = -$ ¼ W is non zero only for the negative arguments. Clearly, therefore,

$f_{X(t_1)}(X_1) \neq f_{X(t_2)}(X_2)$, and the random process $X(t)$ is non stationary.

Example: 3:- Sine wave with Random Phase (Cont...)

$$X(t) = A\cos(2\pi f_c t + \theta)$$

where $\theta \leftarrow$ uniformly distr rv over $[-\pi, \pi]$

Estimate the PSD of $X(t)$. Prove Property 1.

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Use eqn (5)

$$\therefore S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} \cdot d\tau$$

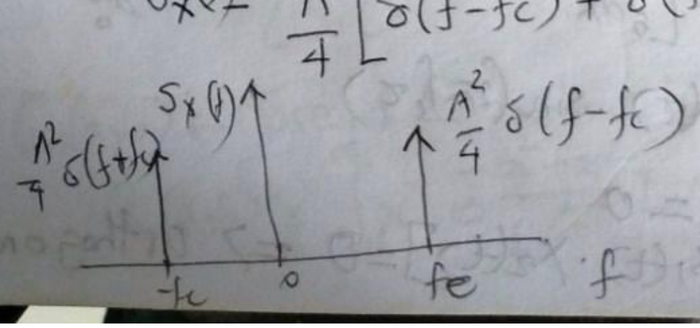
$$= \frac{A^2}{2}\int_{-\infty}^{\infty} \cos(2\pi f_c \tau) \cdot e^{-j2\pi f\tau} \cdot d\tau$$

$$= \frac{A^2}{2}\int_{\infty}^{\infty} R.P\{e^{j2\pi f_c \tau}\} e^{-j2\pi f\tau} \cdot d\tau$$

$$= \frac{A^2}{4}\int_{\infty}^{\infty} (e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau}) e^{-j2\pi f\tau} \cdot d\tau$$

$$= \frac{A^2}{4}\int_{\infty}^{\infty} e^{-j2\pi(f-f_c)\tau} \cdot d\tau + \frac{A^2}{4}\int_{-\infty}^{\infty} e^{-2\pi(f+f_c)\tau} \cdot d\tau$$

$$S_X(f) = \frac{A^2}{4}\left[\delta(f-f_c) + \delta(f+f_c)\right] =$$



Exer 4 : Mixing a Random Process with a Sine Wave

$X(t), \Theta$ ← independent rps

$$Y(t) = X(t) \cdot \cos(2\pi f_c t + \Theta)$$

Show ~~Find~~ PSD of $Y(t)$ is $\frac{1}{4}\left[S_X(f-f_c) + S_X(f+f_c)\right]$

Autocorrelation function of $Y(t)$

$$R_Y(\tau) = E\left[Y(t+\tau) \cdot Y(t)\right]$$

$$= E\left[X(t+\tau)\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\, X(t) \cos(2\pi f_c t +$$

$$= E\left[X(t+\tau) \cdot X(t)\right] \cdot E\left[\right.$$

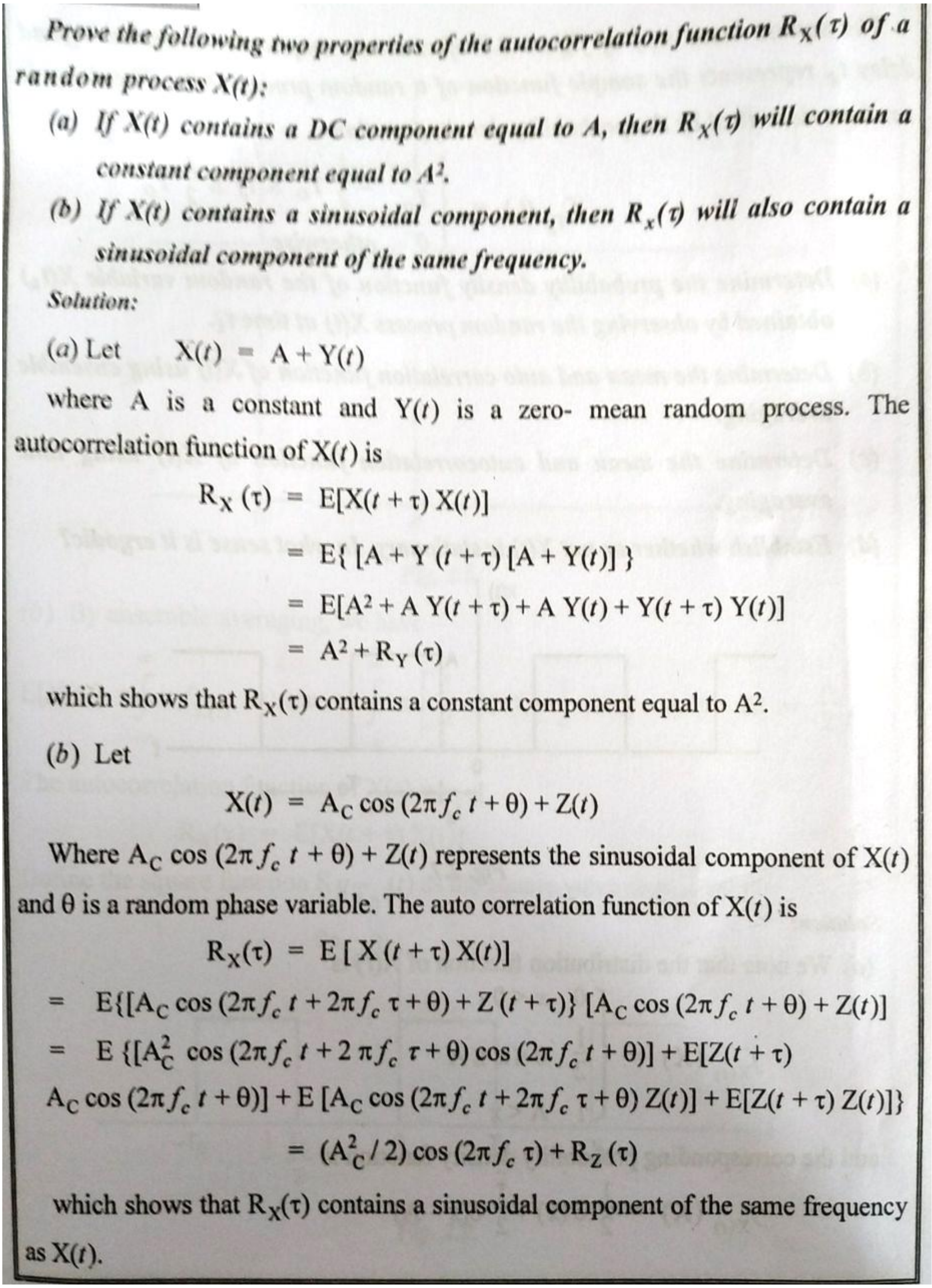
$$= \frac{1}{2} R_X(\tau)\, E\left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)\right]$$

$$= \frac{1}{2} R_X(\tau) \cos 2\pi f_c \tau$$

The PSD is the FT of autocorrelation fn

$$\therefore S_Y(f) = \frac{1}{4}\left[S_X(f-f_c) + S_X(f+f_c)\right]$$

Example -5



Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(a) If $X(t)$ contains a DC component equal to A, then $R_X(\tau)$ will contain a constant component equal to A^2.

(b) If $X(t)$ contains a sinusoidal component, then $R_x(\tau)$ will also contain a sinusoidal component of the same frequency.

Solution:

(*a*) Let $X(t) = A + Y(t)$

where A is a constant and Y(*t*) is a zero- mean random process. The autocorrelation function of X(*t*) is

$$R_X(\tau) = E[X(t+\tau)\,X(t)]$$

$$= E\{\,[A + Y(t+\tau)\,[A + Y(t)]\,\}$$

$$= E[A^2 + A\,Y(t+\tau) + A\,Y(t) + Y(t+\tau)\,Y(t)]$$

$$= A^2 + R_Y(\tau)$$

which shows that $R_X(\tau)$ contains a constant component equal to A^2.

(*b*) Let

$$X(t) = A_C \cos(2\pi f_c\, t + \theta) + Z(t)$$

Where $A_C \cos(2\pi f_c\, t + \theta) + Z(t)$ represents the sinusoidal component of X(*t*) and θ is a random phase variable. The auto correlation function of X(*t*) is

$$R_X(\tau) = E\,[\,X(t+\tau)\,X(t)]$$

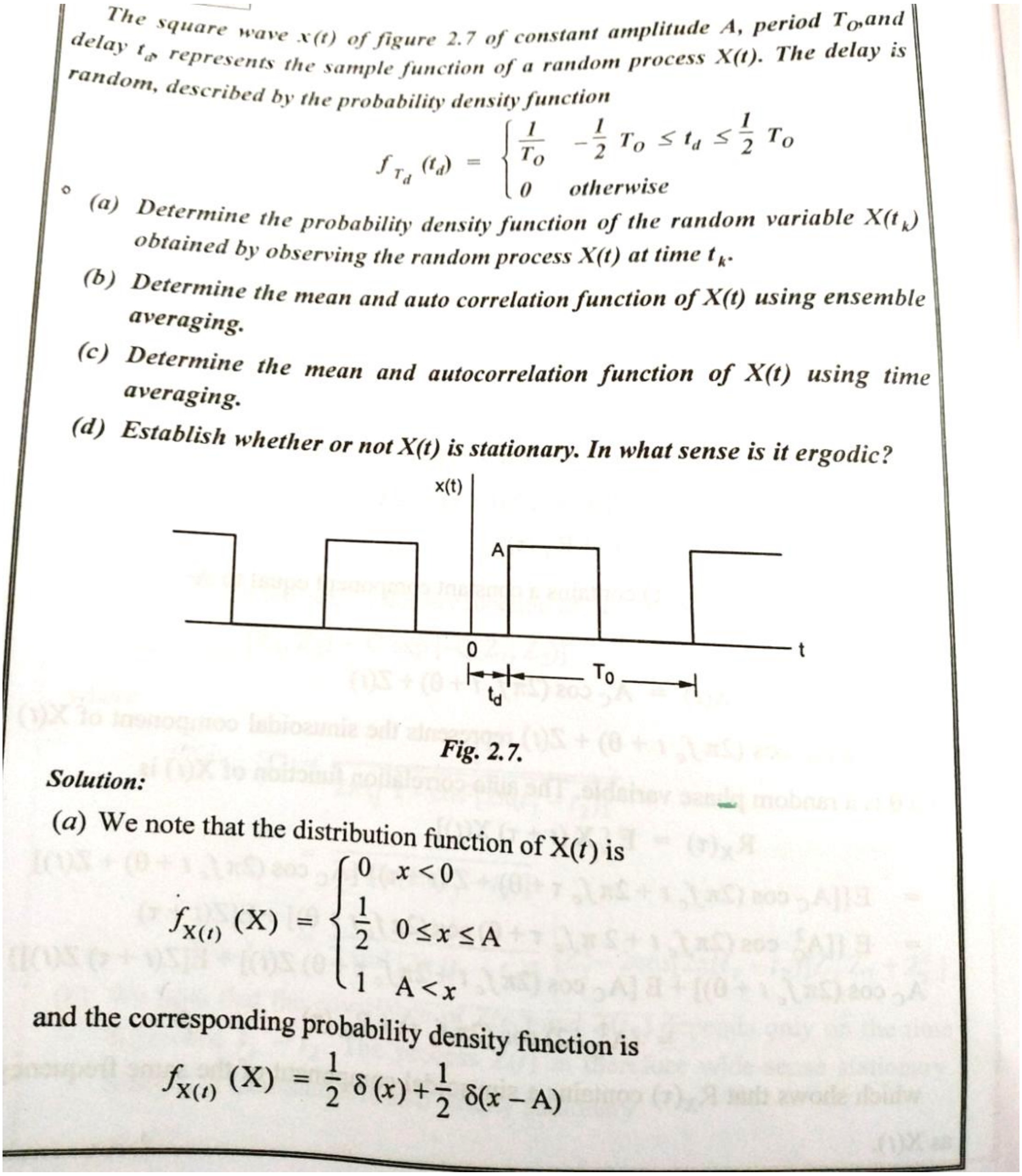
$$= E\{[A_C \cos(2\pi f_c\, t + 2\pi f_c\, \tau + \theta) + Z\,(t+\tau)\}\,[A_C \cos(2\pi f_c\, t + \theta) + Z(t)]$$

$$= E\,\{[A_C^2 \cos(2\pi f_c\, t + 2\,\pi f_c\, \tau + \theta)\cos(2\pi f_c\, t + \theta)] + E[Z(t+\tau)$$
$$A_C \cos(2\pi f_c\, t + \theta)] + E\,[A_C \cos(2\pi f_c\, t + 2\pi f_c\, \tau + \theta)\,Z(t)] + E[Z(t+\tau)\,Z(t)]\}$$

$$= (A_C^2/2)\cos(2\pi f_c\, \tau) + R_Z(\tau)$$

which shows that $R_X(\tau)$ contains a sinusoidal component of the same frequency as X(*t*).

Example -6



The square wave $x(t)$ of figure 2.7 of constant amplitude A, period T_O, and delay t_d, represents the sample function of a random process $X(t)$. The delay is random, described by the probability density function

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T_O} & -\frac{1}{2}T_O \le t_d \le \frac{1}{2}T_O \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the probability density function of the random variable $X(t_k)$ obtained by observing the random process $X(t)$ at time t_k.

(b) Determine the mean and auto correlation function of $X(t)$ using ensemble averaging.

(c) Determine the mean and autocorrelation function of $X(t)$ using time averaging.

(d) Establish whether or not $X(t)$ is stationary. In what sense is it ergodic?

Fig. 2.7.

Solution:

(a) We note that the distribution function of $X(t)$ is

$$f_{X(t)}(X) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x \le A \\ 1 & A < x \end{cases}$$

and the corresponding probability density function is

$$f_{X(t)}(X) = \frac{1}{2}\,\delta(x) + \frac{1}{2}\,\delta(x - A)$$

which are illustrated below:

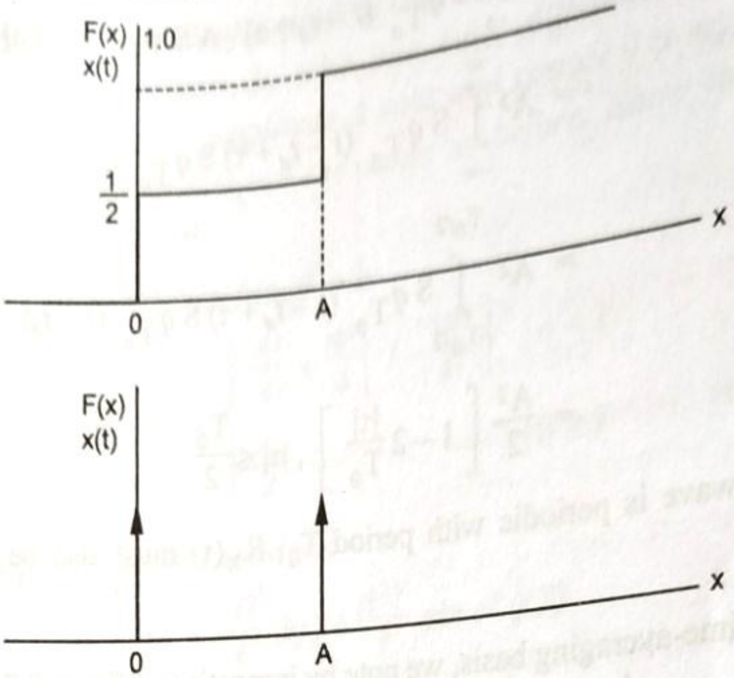


Fig. 2.8.

(*b*) By ensemble averaging, we have

$$E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx = \int_{-\infty}^{\infty} x \left[\frac{1}{2}\, \delta(x) + \frac{1}{2}\, \delta(x - A) \right] dx = \frac{A}{2}$$

The autocorrelation function of X(*t*) is

$$R_X(\tau) = E[X(t + \tau)\, X(t)]$$

Define the square function $Sq_{T_0}(t)$ as the square-wave shown below:

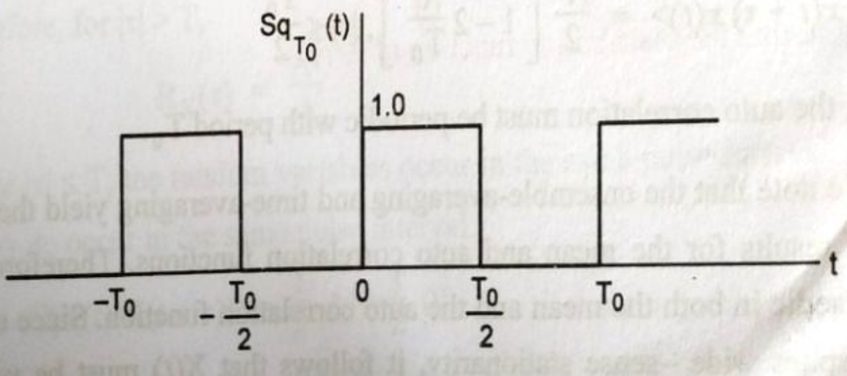


Fig. 2.9.

Then, we may write

$$R_X(\tau) = E[A\, S q_{T_0}(t - t_d + \tau) \cdot A\, S q_{T_0}(t - t_d)]$$

$$= A^2 \int_{-\infty}^{\infty} S q_{T_0}(t - t_d + \tau)\, S q_{T_0}(t - t_d) f_{T_d}(t_d)\, d t_d$$

$$= A^2 \int_{-T_0/2}^{T_0/2} S q_{T_0}(t - t_d + \tau)\, S q_{T_0}(t - t_d) \cdot \frac{1}{T_0}\, d t_d$$

$$= \frac{A^2}{2}\left[1 - 2\frac{|\tau|}{T_0}\right], |\tau| \le \frac{T_0}{2}$$

Since the wave is periodic with period T_0, $R_X(\tau)$ must also be periodic with period T_0.

(*c*) On a time-averaging basis, we note by inspection of figure 2.7 that the mean is $< x(t) > = \frac{A}{2}$

Next, the autocorrelation function

$$< x\,(t + \tau)\, x(t) > = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t + \tau)\, x(t)\, dt$$

has its maximum value of $A^2/2$ at $\tau = 0$, and decreases linearly to zero at $\tau = T_0/2$. Therefore,

$$< x(t + \tau)\, x(t) > = \frac{A^2}{2}\left[1 - 2\frac{|\tau|}{T_0}\right], |\tau| \le \frac{T_0}{2}$$

Again, the auto correlation must be periodic with period T_0.

(*d*) We note that the ensemble-averaging and time-averaging yield the same set of results for the mean and auto correlation functions. Therefore, X(*t*) is ergodic in both the mean and the auto correlation function. Since ergodicity implies wide –sense stationarity, it follows that X(*t*) must be wide-sense stationary.

Example -7

A binary wave consists of a random sequence of symbols 1 and 0, similar to that described in problem 2., with one basic difference: symbol 1 is now represented by a pulse of amplitude A volts and symbol 0 is represented by zero volts. All other parameters are the same as before. Show that for this new random binary wave X(t).

(a) The auto correlation function is

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left[1 - \frac{|\tau|}{T}\right] & |\tau| < T \\ \frac{A^2}{4} & |\tau| \geq T \end{cases}$$

(b) The power spectral density is

$$S_X(f) = \frac{A^2}{4}\,\delta(f) + \frac{A^2T}{4}\,\sin c^2\,(fT)$$

what is the percentage power contained in the DC component of the binary wave?

Solution:

(a) For $|\tau| > T$, the random variables $X(t\ )$ and $X(\ t + \tau)$ occur in different pulse intervals and are therefore independent. Thus,

$$E[X(t)\,X(t+\tau)] = E[X(t)]\,E\,[X(t+\tau)], \qquad |\tau| > T$$

Since both amplitudes are equally likely, we have $E[X(t)] = E\,[x(t+\tau)] = A/2$. Therefore, for $|\tau| > T$,

$$R_X(\tau) = \frac{A^2}{4}$$

For $|\tau| \leq T$, the random variables occur in the same pulse interval if $t_d < T - |\tau|$. If they do occur in the same pulse interval.

$$E[X(t)\,X(t+\tau)] = \frac{1}{2}\,A^2 + \frac{1}{2}\,0^2$$

$$= \frac{A^2}{2}$$

We thus have a conditional expectation:

$$E[X(t)\,X(t+\tau)] = A^2/2,\ t_d < T - |\tau|$$

$$= A^2/4, \text{ otherwise}$$

Averaging over t_d, we get

$$R_X(\tau) = \int_0^{T-|\tau|} \frac{A2}{2T}\, d\,t_d + \int_{T-|\tau|}^{T} \frac{A^2}{4T}\, d\,t_d$$

$$= \frac{A^2}{4}\left[1 - \frac{|\tau|}{T}\right] + \frac{A^2}{4},\ |\tau| \le T$$

(*b*) The power spectral density is the Fourier transform of the autocorrelation function. The Fourier transform of

$$g(\tau) = 1 - \frac{|\tau|}{T},\ |\tau| \le T$$

$$= 0, \text{ otherwise,}$$

is given by

$$G(f) = T\,\text{sinc}^2\,(fT)$$

Therefore,

$$S_X(f) = \frac{A^2}{4}\,\delta(f) + \frac{A^2T}{4}\,\sin c^2\,(fT).$$

We next note that

$$\frac{A^2}{4}\int_{-\infty}^{\infty} \delta(f)\,df = \frac{A^2}{4},$$

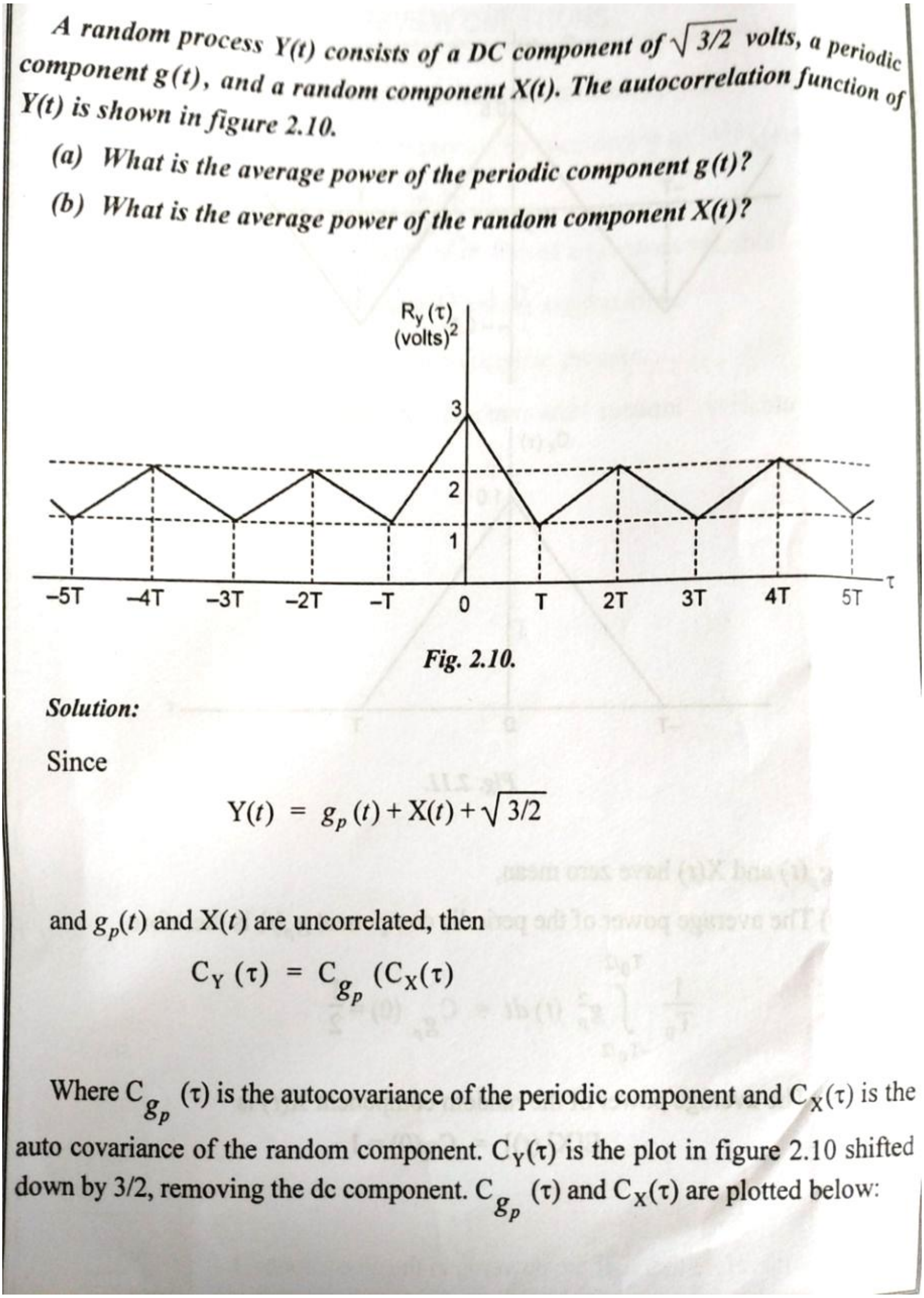
$$\frac{A^2}{4}\int_{-\infty}^{\infty} T\text{sinc}^2(fT)\,df = \frac{A^2}{4},$$

$$\int_{-\infty}^{\infty} S_X(f)\,d\,f = R_X(0)$$

$$= \frac{A^2}{2}.$$

It follows therefore that half the power is in the dc component.

Example -8



A random process Y(t) consists of a DC component of $\sqrt{3/2}$ volts, a periodic component g(t), and a random component X(t). The autocorrelation function of Y(t) is shown in figure 2.10.

(a) What is the average power of the periodic component g(t)?

(b) What is the average power of the random component X(t)?

Fig. 2.10.

Solution:

Since

$$Y(t) = g_p(t) + X(t) + \sqrt{3/2}$$

and $g_p(t)$ and X(t) are uncorrelated, then

$$C_Y(\tau) = C_{g_p}\ (C_X(\tau)$$

Where C_{g_p} (τ) is the autocovariance of the periodic component and $C_X(\tau)$ is the auto covariance of the random component. $C_Y(\tau)$ is the plot in figure 2.10 shifted down by 3/2, removing the dc component. C_{g_p} (τ) and $C_X(\tau)$ are plotted below:

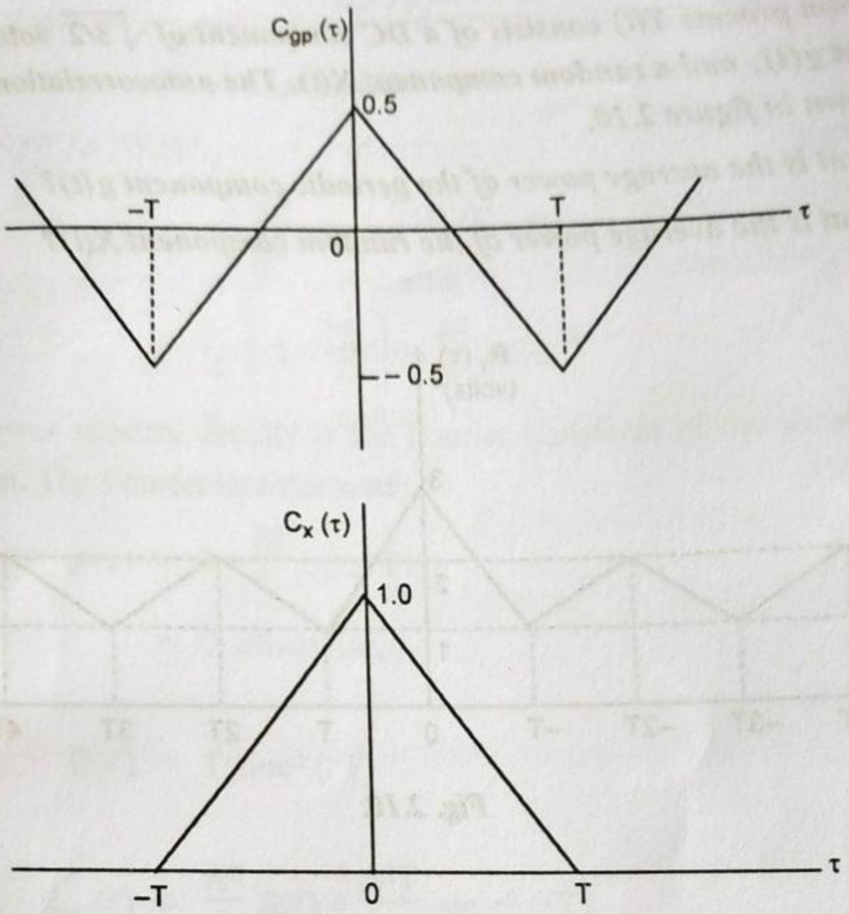


Fig. 2.11.

Both $g_p(t)$ and X(t) have zero mean,

(a) The average power of the periodic component $g_p(t)$ is therefore,

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p^2\,(t)\,dt = C_{g_p}\,(0) = \frac{1}{2}$$

(b) The average power of the random component X(t) is

$$E[X^2\,(t)] = C_X(0) = 1$$

Pulse Code Modulation (PCM)

EC 302 DIGITAL COMMUNICATION
MODULE – I
PART-2

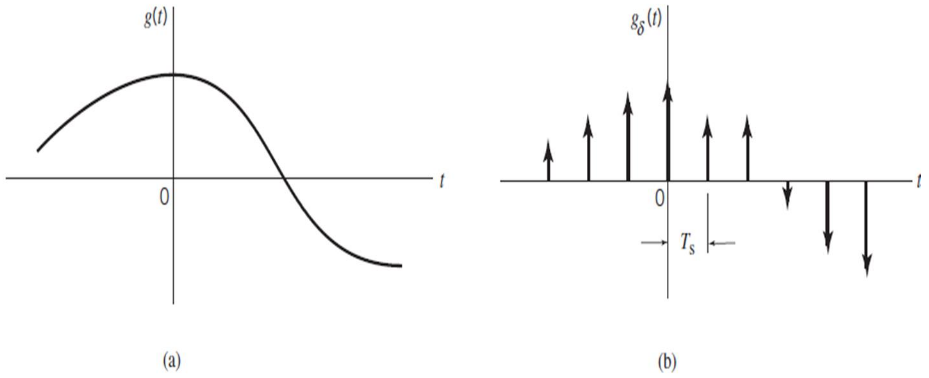
Pulse Modulation

- In Continuous-Wave (CW) Modulation, some parameter of a sinusoidal carrier wave is varied continuously in accordance with the message signal.
- In Pulse Modulation, some parameter of a pulse train is varied in accordance with the message signal. Two types →
  - **1.** Analog pulse modulation, in which a periodic pulse train is used as the carrier wave and some characteristic of each pulse (e.g., amplitude, duration, or position) is varied in a continuous manner in accordance with the corresponding sample value of the message signal.
  - Thus, in analog pulse modulation, information is transmitted basically in analog form but the transmission takes place at discrete times.
  - **2.** Digital pulse modulation, in which the message signal is represented in a form that is discrete in both time and amplitude, thereby permitting transmission of the message in digital form as a sequence of coded pulses

Sampling Process

- The Sampling process is analyzed in the time domain.
- It is an operation that is basic to Digital signal processing and digital communications.
- The sampling process enables an Analog signal to be converted into a corresponding sequence of Samples that are usually spaced uniformly in time.
- It is mandatory to choose the Sampling rate properly in relation to the Bandwidth of the message signal, so that the Sequence of Samples uniquely defines the original Analog signal.

- Consider an arbitrary signal $g(t)$ of finite energy, which is specified for all time t.
- A segment of the signal $g(t)$ is shown.
- Suppose that we sample the signal $g(t)$ instantaneously and at a uniform rate, once every T_s seconds.
- Obtain an infinite sequence of samples spaced T_s seconds apart and denoted by $\{g(nT_s)\}$, where n takes on all possible integer values, positive as well as negative.
- We refer to T_s as the sampling period, and to its reciprocal $f_s = 1/T_s$ as the sampling rate.
- this ideal form of sampling is called Instantaneous Sampling.



The sampling process.
(a) Analog signal.
(b) Instantaneously sampled version of the Analog signal.

- Let $g_\delta(t)$ denote the signal obtained by individually weighting the elements of a periodic sequence of delta functions spaced T_s seconds apart by the sequence of numbers $\{g(nT_s)\}$.

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) \qquad \text{................(1)}$$

- refer to $g_\delta(t)$ as the *ideal sampled signal*.
- The term $\delta\ (t - nTs)$ represents a delta function positioned at time $t = nT_s$.

- A delta function weighted in this manner is closely approximated by a rectangular pulse of duration Δt and amplitude $g(nTs)/\Delta t$.
- the smaller Δ t the better the approximation will be.

$$\therefore \; g_\delta(t) \rightleftharpoons f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \quad \text{................(2)}$$

- where $G(f)$ is the Fourier transform of the original signal $g(t)$ and f_s is the sampling rate.

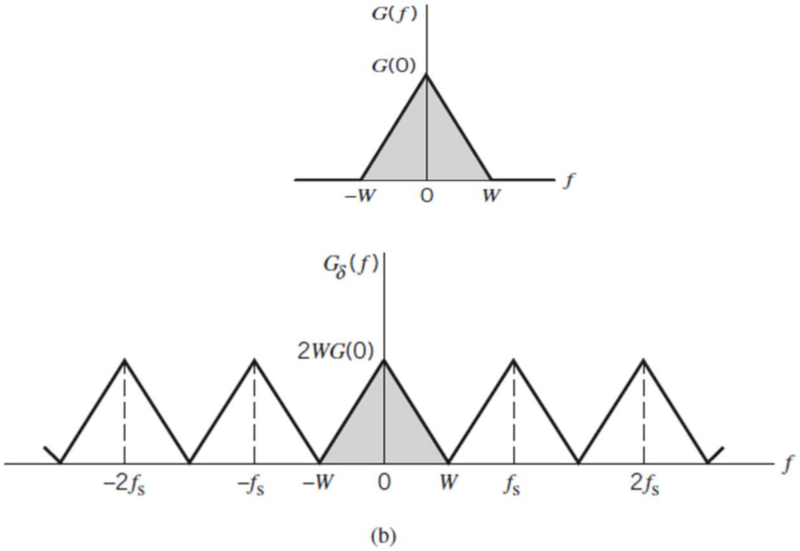
- The process of uniformly sampling a continuous-time signal of finite energy results in a periodic spectrum with a frequency equal to the sampling rate.
- Take the Fourier transform of both sides of eqn (1) and note that the Fourier transform of delta function $\delta(t - nTs)$ is equal to $e^{-j2\pi nfTs}$
- Let $G_\delta(f)$ denote Fourier transform of $g_\delta(t)$,

$$\therefore \; G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nfT_s) \quad \text{................(3)}$$

- This describes the discrete-time Fourier transform.
- It may be viewed as a complex Fourier series representation of the periodic frequency function $G_\delta(f)$, with the sequence of samples $\{g(nT_s)\}$ defining the coefficients of the expansion.

Band limited signal

- Let the signal *g*(*t*) be strictly band limited, with no frequency components higher than *W* hertz, as shown.
- the Fourier transform *G*(*f*) of the signal *g*(*t*) has the property *G*(*f*) is zero for | *f* | ≥ *W*
- choose the sampling period $T_s = 1/2W$.
- Then the corresponding spectrum G_δ (*f*) of the sampled signal G_δ (*t*) is as shown .



(a) Spectrum of a strictly band-limited signal *g*(*t*).
(b) Spectrum of the sampled version of *g*(*t*) for a sampling period *T*s = 1/2*W*.

- Putting $T_s = 1/2W$ in eqn (3) yields,

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right) \quad \text{................(4)}$$

- Isolating the term on rhs of eqn(2), agreeing to $m = 0$, the Fourier transform of $G_\delta\ (t)$ may also be expressed as,

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - mf_s) \quad \text{................(5)}$$

- impose the following two conditions:
  1. $G(f) = 0$ for $|f| \geq W$.
  2. $f_s = 2W$.

$$\therefore G(f) = \frac{1}{2W}G_\delta(f), \qquad -W < f < W \quad \text{................(6)}$$

- Use (4) into (6),

$$G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

................(7)

- Eqn (7) is the needed formula for the frequency-domain description of Sampling.
- If the sample values *g*(*n*/2*W*) of the signal *g*(*t*) are specified for all *n*, then the Fourier transform *G*(*f*) of that signal is uniquely determined.
- As *g*(*t*) is related to *G*(*f*) by the inverse Fourier transform, *g*(*t*) is itself uniquely determined by sample values *g*(*n*/2*W*) for all values of *n*.
- So, the sequence {*g*(*n*/2*W*)} has all the information contained in the original signal *g*(*t*).

- Consider next the problem of reconstructing the signal *g*(*t*) from the sequence of sample values {*g*(*n*/2*W*)}.
- Use (7) in the formula for the IFT,

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\, df$$

$$\therefore g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right] df$$

…………….(8)

$$\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right] df = \frac{\sin(2\pi Wt - n\pi)}{2\pi Wt - n\pi}$$

$$= \operatorname{sinc}(2Wt - n)$$

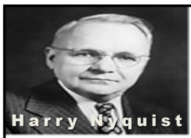
So, (8) reduces to the infinite-series expansion,

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\operatorname{sinc}(2Wt-n), \qquad -\infty < t < \infty$$

…………….(9)

Eqn (9) is the desired *reconstruction formula.*

- This formula provides the basis for reconstructing the original signal *g*(*t*) from the sequence of sample values {*g*(*n*/2*W*)}, with sinc(2*Wt*) playing the role of a *basis function* of the expansion.
- Each sample, *g*(*n*/2*W*), is multiplied by a delayed version of *basis function*, sinc(2*Wt* − *n*), and all the resulting individual waveforms in the expansion are added to reconstruct the original signal *g*(*t*).



The Sampling Theorem

1. A band-limited signal of finite energy that has no frequency components higher than *W* hertz is completely described by specifying the values of the signal instants of time separated by 1/2*W* seconds. *Frequency-Domain Description of Sampling*

2. A band-limited signal of finite energy that has no frequency components higher than *W* hertz is completely recovered from a knowledge of its samples taken at the rate of 2*W* samples per second. *Reconstruction formula*

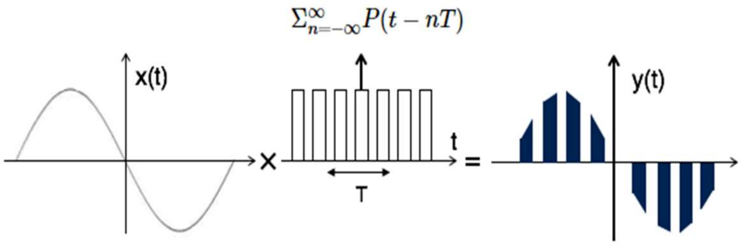
- Part 1 of the Sampling theorem, following from eqn (7), is performed in the Transmitter.
- Part 2 of the Sampling theorem, following from eqn (9), is performed in the Receiver.
- For a signal bandwidth of *W* hertz, the sampling rate of 2*W* samples per second is called the Nyquist rate.
- Its reciprocal 1/2*W* (measured in seconds) is called the Nyquist interval.

Sampling Techniques

- Impulse sampling or Instantaneous sampling.
  - Use of Impulse Train as already studied!
- Natural sampling.
  - Use of Pulse Train
- Flat Top sampling.
  - Use of Pulse Train plus Pulse levelling

Natural sampling

- Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. multiply input signal x(t) with pulse train as shown below,



- The output of sampler is,

y(t)= x(t) × p(t)

$$= x(t) \times \Sigma_{n=-\infty}^{\infty} P(t - nT)$$ ....(10)

- The exponential Fourier series representation of p(t) can be given as,

$$p(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_s t}$$ ....(11)

$$F_n = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt = \frac{1}{TP}(n\omega_s)$$

$$\therefore p(t) = \Sigma_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$ ....(12)

$$y(t) = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}$$ ....(13)

$$F.T\,[y(t)] = F.T[\frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, x(t)\, e^{jn\omega_s t}]$$

$$= \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, F.T\,[x(t)\, e^{jn\omega_s t}]$$

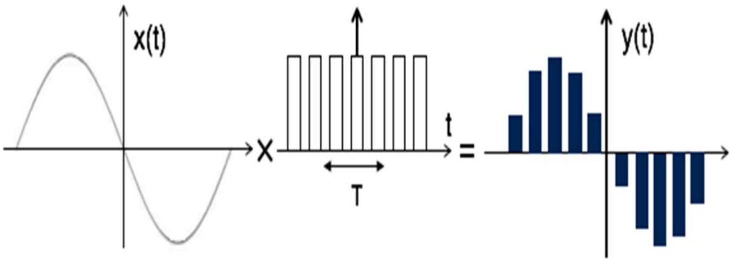
$$F.T\,[x(t)\, e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore Y[\omega] = \frac{1}{T} \Sigma_{n=-\infty}^{\infty} P(n\omega_s)\, X[\omega - n\omega_s]$$

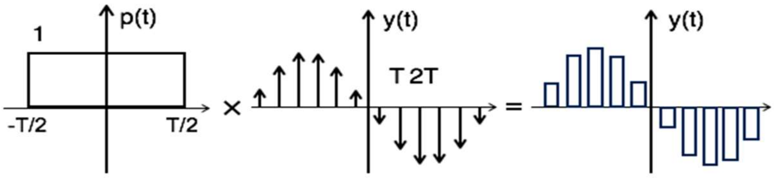
................(14)

Flat Top Sampling

- During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top.
- Here, the top of the samples are flat i.e. they have constant amplitude.
- Hence, it is called as flat top sampling or practical sampling.
- Flat top sampling makes use of sample and hold circuit.



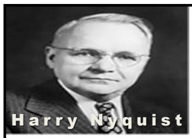
- Theoretically, the sampled signal can be obtained by convolution of rectangular pulse p(t) with ideally sampled signal say $y_\delta(t)$ as shown in the diagram:



i.e. $y(t) = p(t) \times y_\delta(t)$ ....(15)

$$Y[\omega] = F.T\,[P(t) \times y_\delta(t)]$$

$Y[\omega] = P(\omega)\,Y_\delta(\omega)$ ....(16)



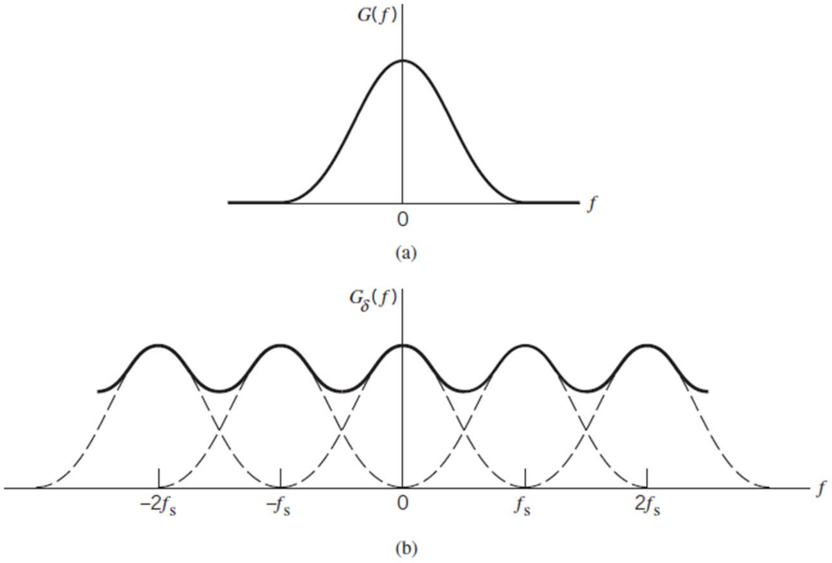
The Sampling Theorem

1. A band-limited signal of finite energy that has no frequency components higher than *W* hertz is completely described by specifying the values of the signal instants of time separated by 1/2*W* seconds. *Frequency-Domain Description of Sampling*

2. A band-limited signal of finite energy that has no frequency components higher than *W* hertz is completely recovered from a knowledge of its samples taken at the rate of 2*W* samples per second. *Reconstruction formula*

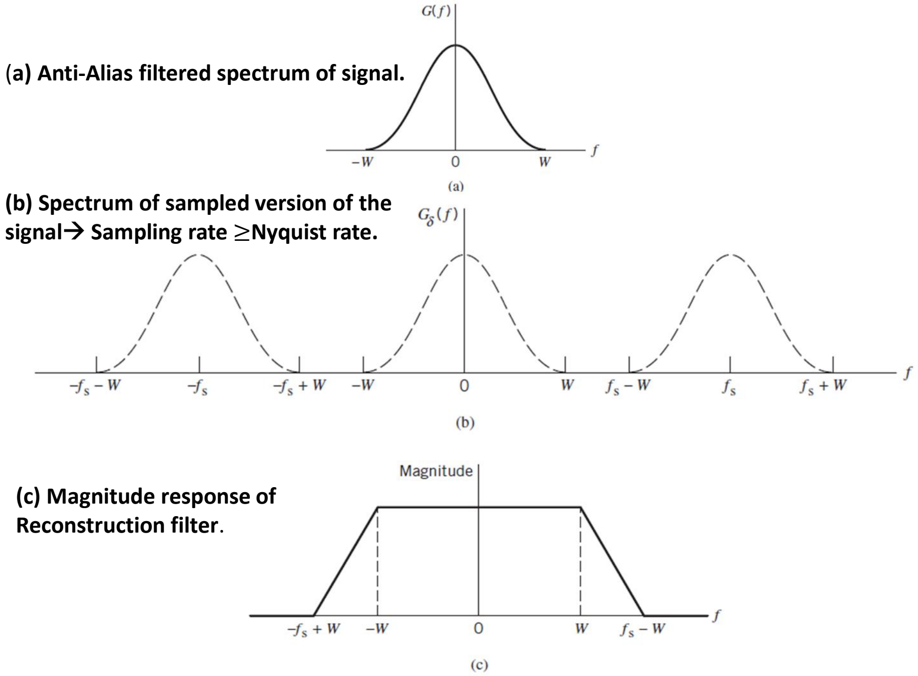
Aliasing Phenomenon

- Derivation of the sampling theorem is based on the assumption that the signal $g(t)$ is strictly band limited.
- In practice, however, a message signal is not strictly band limited.
- So some degree of under-sampling happens, as a consequence of which Aliasing is produced by the sampling process.
- Aliasing refers to the phenomenon of a high-frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.



(a) Spectrum of a signal.
(b) Spectrum of an under-sampled version of the signal exhibiting the aliasing phenomenon.

- To combat the effects of Aliasing , there are two corrective measures:
  - **1.** Prior to sampling, a low-pass Anti-Aliasing filter is used to attenuate those high frequency components of the signal that are not essential to the information being conveyed by the message signal $g(t)$.
  - **2.** The filtered signal is sampled at a rate slightly higher than the Nyquist rate.
- The use of a Sampling rate higher than the Nyquist rate also has the benefit of simplifying the design of the Reconstruction filter used to recover the original signal from its sampled version.



(a) Anti-Alias filtered spectrum of signal.

(b) Spectrum of sampled version of the signal→ Sampling rate ≥Nyquist rate.

(c) Magnitude response of Reconstruction filter.

- Design of the Reconstruction filter may be specified as follows:

  ❑ The Reconstruction filter is low-pass with a pass band extending from –*W* to *W*, which is itself determined by the Anti-Aliasing filter.

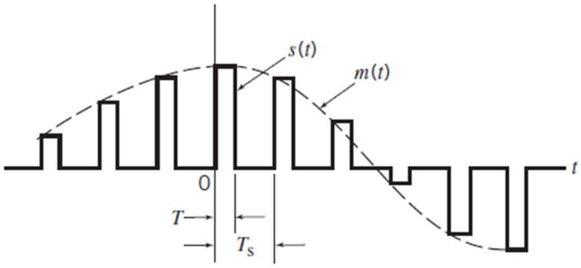
  ❑ The Reconstruction filter has a transition band extending (for positive frequencies) from *W* to (f_s – *W*), where f_s is the Sampling rate.

EXAMPLE-1-**Sampling of Voice Signals**

- The frequency band, from 100 Hz to 3.1 kHz, is adequate for Telephonic communication.
- This limited frequency band is accomplished by passing the voice signal through a Low-Pass Filter with its cutoff frequency set at 3.1 kHz.
- Such a Filter may be viewed as an Anti-Aliasing Filter.
- With such a cutoff frequency, the Nyquist rate is f_s = 2 x 3.1 = 6.2 kHz.
- The standard Sampling rate for the waveform coding of voice signals is 8 kHz.
- Design specifications for the Reconstruction (Low-Pass) Filter in the Receiver are as follows:
  - Cut-off frequency 3.1 kHz
  - Transition band 6.2 to 8 kHz
  - Transition-band width 1.8 kHz.

Pulse-Amplitude Modulation

- PAM is a linear modulation process where the amplitudes of regularly spaced pulses are varied in proportion to the corresponding sample values of a continuous message signal.



Flat-top samples, representing an analog signal.

- There are two operations involved in the generation of the PAM signal:
  - 1. *Instantaneous sampling* of the message signal $m(t)$ every T_s seconds, where the sampling rate $f_s = 1/T_s$ is chosen in accordance with the sampling theorem.
  - 2. *Lengthening* the duration of each sample so obtained to some constant value T.
- These two operations are jointly referred to as **Sample and Hold**.
- Lengthen the duration of each sample to avoid the use of an excessive channel bandwidth, because bandwidth is inversely proportional to pulse duration.

- Let $s(t)$ denote the sequence of flat-top pulses.
- Express PAM signal as discrete convolution sum,

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t-nT_s) \qquad \ldots\ldots\ldots\ldots\ldots.(17)$$

- where T_s is the *sampling period* and $m(nT_s)$ is the sample value of $m(t)$ obtained at time $t = nT_s$.
- The $h(t)$ is a Fourier-transformal pulse.

$$h(t-nT_s) = \int_{-\infty}^{\infty} h(t-\tau)\delta(t-nT_s)\,\mathrm{d}\tau \qquad \ldots\ldots\ldots\ldots\ldots.(18)$$

$$\therefore\; s(t) = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} m(nT_s)\delta(t-nT_s)\right]h(t-\tau)\,\mathrm{d}\tau \qquad \ldots\ldots\ldots\ldots\ldots.(19)$$

- Spot that the expression inside the brackets in eqn (19) is the instantaneously sampled version of the message signal *m*(*t*),

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t-nT_s) \quad \text{................(20)}$$

$$\therefore \; s(t) = \int_{-\infty}^{\infty} m_\delta(t)h(t-\tau)\,\mathrm{d}\tau \quad = m_\delta(t) \star h(t) \quad \text{................(21)}$$

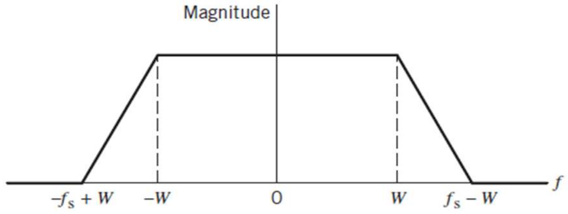
- Take Fourier transform of both sides of (21),

$$\therefore S(f) = M_\delta(f)H(f) \quad \text{................(22)}$$

$$M_\delta(f) = f_s \sum_{k=-\infty}^{\infty} M(f-kf_s) \quad \text{................(23)}$$

$$\therefore \; S(f) = f_s \sum_{k=-\infty}^{\infty} M(f-kf_s)H(f) \quad \text{................(24)}$$

- Given this formula in eqn(24), how to recover the original message signal *m*(*t*)?
- As a first step in this reconstruction, pass *s*(*t*) thro' a LPF whose frequency response is shown.
- It is assumed that the message signal is limited to bandwidth *W* and the sampling rate $f_s \geq$ Nyquist rate 2*W*.
- The spectrum of the resulting filter output is equal to *M*(*f*)*H*(*f*).
- This output is equivalent to passing the original message signal *m*(*t*) through another LPF of frequency response *H*(*f*).



Magnitude response of Reconstruction filter.

Practical Considerations for PAM

- The transmission of a PAM signal imposes rather rigorous requirements on the frequency response of the channel, because of the relatively short duration of the transmitted pulses.
- To rely on amplitude as the parameter subject to modulation, means the noise performance of a PAM system can never be better than baseband signal transmission.
- In practice, for transmission over a channel, PAM is used only as the initial means of message processing, then the PAM signal is changed to some other more apt form of pulse modulation.

Transmission over Baseband Channel

Module-II Part-1

EC302 Digital Communication

Introduction

- In baseband signaling, the received waveforms are ideally in a pulse-like form.
- Alas! The arriving baseband pulses are not in the form of ideal pulse shape, each one occupying its own symbol interval.
- The filtering at the Transmitter and the Channel, cause the received pulse sequence to “suffer” from Inter Symbol Interference (ISI) and thus appear as an amorphous “smeared” signal, not quite ready for sampling and detection.
- The goal of the demodulator is to recover a baseband pulse with the best possible signal-to-noise ratio (SNR), free of any ISI.
- Equalization is a technique use to accomplish this goal, which embodies a sophisticated set of signal-processing techniques, making it possible to compensate for channel-induced interference.

- The task of detector is to retrieve the bit stream from the corrupted received waveform, affected by signal distortions, channel impairments and noise.
- There are two causes for error-performance degradation.
- The first is the effect of Filtering at the Transmitter, Channel and Receiver.
- A non-ideal system transfer function causes symbol "Smearing" or Inter Symbol Interference (ISI).
- Another more significant cause for error-performance degradation is Electrical Noise and interference produced by additive thermal noise in amplifiers and circuits, atmospheric noise, switching transients, intermodulation noise as well as interfering signals from other sources.

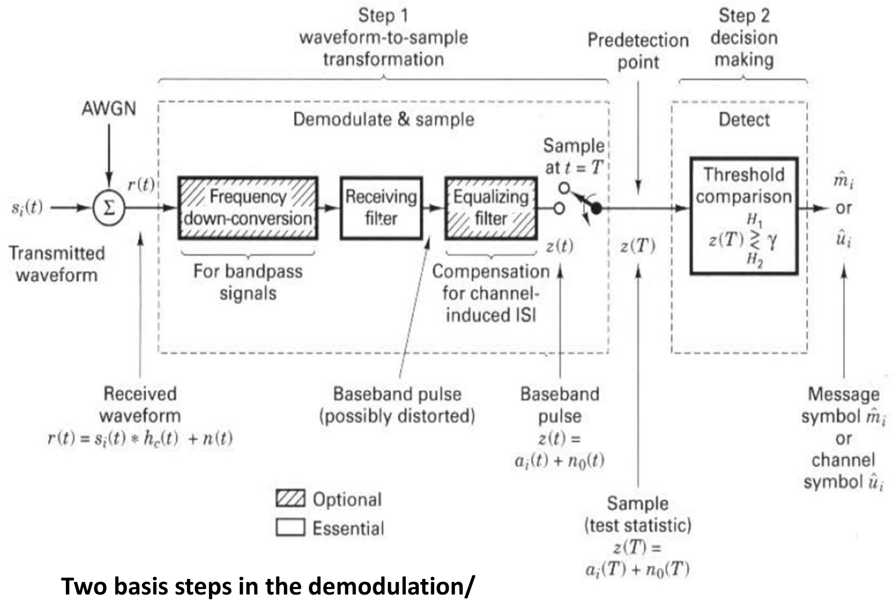
- Noise amplitude of Thermal noise is distributed according to a normal or Gaussian distribution.
- The PSD of white noise is $G_n(f)=N_0/2$ and is flat for all frequencies of interest (low frequencies up to a frequency of 10^{12}Hz).
- Due to this constant PSD, it is referred to as White Noise.
- Hence the "deadly" AWGN is used to model the noise in the detection process and in design of receivers.
- A channel that infects the information bearing signal in this manner is designated as AWGN channel.

The Matched Filter

- A Matched Filter is a linear Filter designated to provide the <u>Maximum Signal-to-Noise Powe</u>r ratio at its Output for a given Transmitted Symbol Waveform.
- Consider that a known Signal s(t) plus AWGN n(t) is the Input to a LTI Receiver Filter followed by a Sampler

→as shown in the diagram to follow…

From book on DC by esteemed pioneer BERNARD SKLAR



Two basis steps in the demodulation/ detection of digital signals

Analysis of Matched Filter

- At time t=T, the sampler output z(T) consists of a signal component a_i and a noise component n_0.
- The <u>variance</u> of the output noise is denoted as σ_0^2, so that the ratio of instantaneous signal power to the average noise power, at time t=T, out of the sampler in *step 1* is,

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} \quad \text{................(1)}$$

- The aim is to find the <u>Filter Transfer Function</u> $H_0(f)$ that Maximizes eqn(1).

- Express the signal $a_i(t)$ at the filter output in terms of the filter transfer function H(f) and the Fourier transform of input signal, as

$$a_i(t) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi ft}\, df \quad \text{................(2)}$$

- The PSD of white noise is $G_n(f)=N_0/2$ and is flat for all frequencies of interest (up to a frequency of 10^{12}Hz).
- So the output noise power can be expressed as ,

$$\sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df \text{................(3)}$$

- From eqn(1),

$$\left(\frac{S}{N}\right)_T = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}\,df\right|^2}{N_0/2\int_{-\infty}^{\infty}|H(f)|^2\,df}$$ …………….(4)

- Next evaluate the value of H(f) = H_0(f) for which the maximum S/N is achieved, by using Schwarz’s inequality as shown below,

$$\left|\int_{-\infty}^{\infty} f_1(x)f_2(x)\,dx\right|^2 \le \int_{-\infty}^{\infty}|f_1(x)|^2\,dx\int_{-\infty}^{\infty}|f_2(x)|^2\,dx$$ …………….(5)

- The equality holds if $f_1(x) = kf_2^*(x)$, where k← arbitrary constant, *← complex conjugate

- Replace, $f_1(x)$ with H(f) and $f_2(x)$ with S(f)e $^{j2\pi ft}$,

$$\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}df\right|^2 \le \int_{-\infty}^{\infty}|H(f)|^2\,df\int_{-\infty}^{\infty}|S(f)|^2\,df$$ …………….(6)

- Substituting into eqn(1),

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2\,df$$ …………….(7)

$$\max\left(\frac{S}{N}\right)_T = \frac{2E}{N_0}$$ …………….(8)

- Where the energy E of the input signal s(t) is,

$$E = \int_{-\infty}^{\infty}|S(f)|^2\,df$$ …………….(9)

- Thus, the maximum output $(S/N)_T$ depends on the input signal energy and the PSD of the noise, not on the particular shape of the waveform that is used.
- The equality of eqn (6) holds only if the optimum filter transfer function $H_0(f)$ is employed, such that,

$$H(f) = H_0(f) = kS^*(f)e^{-j2\pi fT} \quad \text{…………….(10)}$$

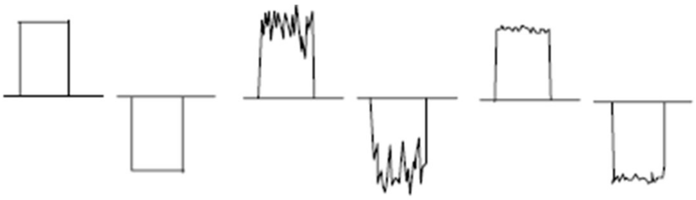
$$h(t) = \mathcal{F}^{-1}\{kS^*(f)e^{-j2\pi fT}\} \quad \text{…………….(11)}$$

- Since s(t) is a real-valued signal,

$$h(t) = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \quad \text{…………….(12)}$$

- Therefore, the impulse response of a Filter that produced the Maximum output Signal-to-Noise ratio is the mirror image of the message signal s(t), delayed by the symbol time duration T.
- The delay of T seconds makes eqn(12) causal, i.e., the delay of T seconds makes h(t) a function of positive time in the interval $0 \le t \le T$.
- Without the delay of T seconds, the response s(-t) is unrealizable, since it describes a response as a function of negative time.

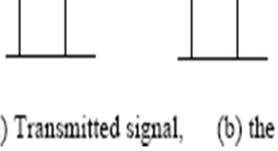
- Input signal s(t)+n(t)



(a) Transmitted signal, square pulses
(b) At the receiver, distorted by a lot of noise.
(c) After the receive filter, looking a good deal more like the transmitted signal.

Matched Filter

The signal to noise ratio is maximized when the impulse response of that filter is exactly a reversed and time delayed copy of the transmitted signal.

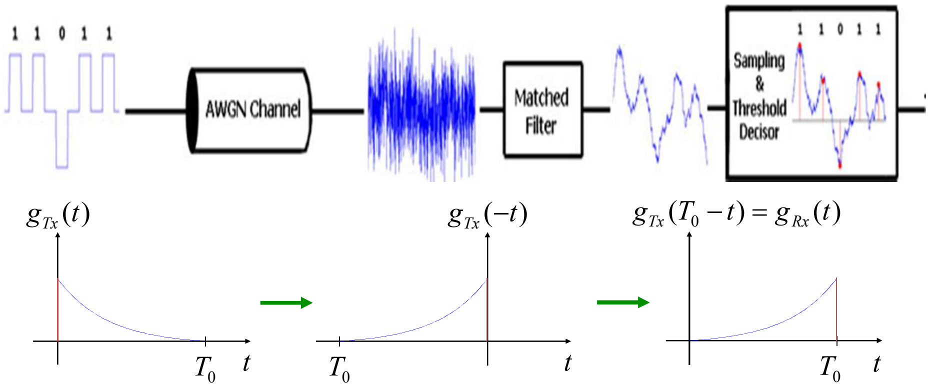


(a) Transmitted signal, (b) the required impulse response of the receive filter.

Matched filter example

- Received SNR is maximized at time T_0

Matched Filter: optimal receive filter for maximized $\frac{S}{N}$



Transmit filter

Receive filter (matched filter)

Inference

- The impulse response of the optimum filter is a **scaled, time reversed and delayed version** of the input signal g(t) i.e., it is matched to the input signal.
- A linear time invariant filter defined in this way is called a **Matched filter.**

Properties of Matched Filter

- A filter that is matched to a pulse signal g(t) of duration T, is characterized by an impulse response that is a time reversed and delayed version of the input.
- The peak pulse signal-to-noise ratio of a matched filter depends only on the ratio of the signal energy to the PSD of the white noise at the filter input.
- The **Rayleigh's energy theorem** states that the integral of the squared magnitude spectrum of a pulse signal w.r.t frequency is equal to the signal energy.

Properties of Matched Filters

Consider a known signal $g(t)$,

$$
\begin{aligned}
G_0(f) &= H_{\text{opt}}(f)G(f) \\
&= kG^*(f)G(f)\exp(-j2\pi fT) \\
&= k|G(f)|^2\exp(-j2\pi fT)
\end{aligned}
$$

$$
\begin{aligned}
g_0(T) &= \int_{-\infty}^{\infty} G_0(f)\exp(j2\pi fT)\,df \\
&= k\int_{-\infty}^{\infty}|G(f)|^2\,df
\end{aligned}
$$

$$g_0(T) = kE$$

$$
\begin{aligned}
E\left[n^2(t)\right] &= \int_{-\infty}^{\infty} S_N(f)\,df \\
&= \frac{k^2N_0}{2}\int_{-\infty}^{\infty}|G(f)|^2\,df \\
&= k^2N_0E/2
\end{aligned}
$$

$$\eta_{\max} = \frac{(kE)^2}{(kN_0E/2)} = \frac{2E}{N_0}$$

which is independent of waveform (E/N_0 = signal energy to noise PSD ratio)

► **EXAMPLE** **Matched Filter for Rectangular Pulse**

Consider a signal $g(t)$ in the form of a rectangular pulse of amplitude A and duration T, as shown in Figure . In this example, the impulse response $h(t)$ of the matched filter has exactly the same waveform as the signal itself. The output signal $g_o(t)$ of the matched filter produced in response to the input signal $g(t)$ has a triangular waveform, as shown in Figure

The maximum value of the output signal $g_o(t)$ is equal to kA^2T, which is the energy of the input signal $g(t)$ scaled by the factor k; this maximum value occurs at $t = T$, as indicated in Figure

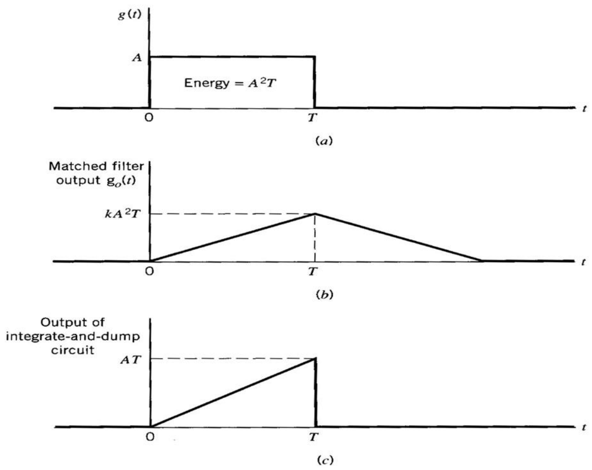
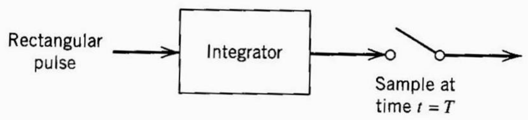


FIGURE (a) Rectangular pulse. (b) Matched filter output. (c) Integrator output.

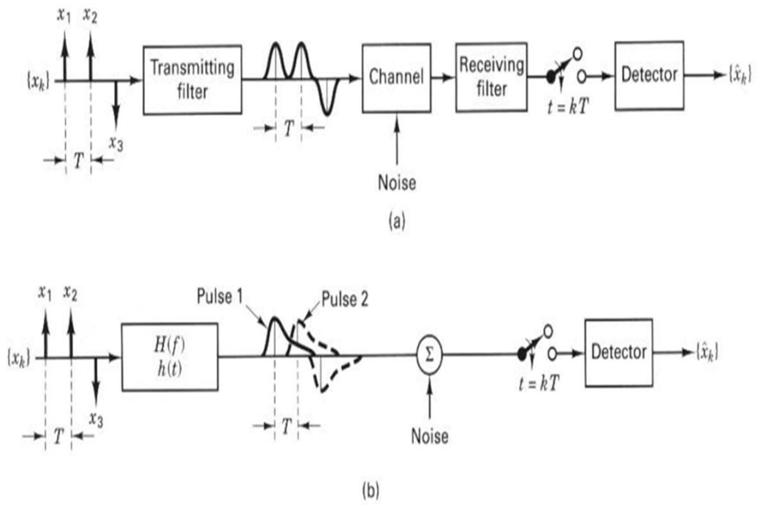


Integrate-and-dump circuit.

For the special case of a rectangular pulse, the matched filter may be implemented using a circuit known as the *integrate-and-dump circuit*, a block diagram of which is shown in Figure The integrator computes the area under the rectangular pulse, and the resulting output is then sampled at time $t = T$, where T is the duration of the pulse. Immediately after $t = T$, the integrator is restored to its initial condition; hence the name of the circuit. Figure shows the output waveform of the integrate-and-dump circuit for the rectangular pulse of Figure We see that for $0 \leq t \leq T$, the output of this circuit has the *same waveform* as that appearing at the output of the matched filter; the difference in the notations used to describe their peak values is of no practical significance. ◄

INTERSYMBOL INTERFERENCE

- There are various filters in the transmitter, receiver and in the channel.
- At the transmitter, the information symbols, characterized as impulse or voltage levels modulate pulses that are then filtered to comply with some band width constraint.
- For baseband signals, the wired channel has distributed reactance that distorts the pulses.
- Some band pass systems, such as wireless systems are characterized by fading channels that behave like undesirable filters manifesting signal distortion.
- When the receiving filter is configured to compensate for the distortion caused by both the transmitter and channel, it is referred to as an equalizing filter or receiving/equalizing filter.
- A convenient model for the system is to lump all the filtering effects into one overall equivalent system transfer function.



Intersymbol Interference in the detection process
(a) Typical baseband digital system **(b) Equivalent model**

- $H_t(f)$ characterizes the transmitting filter, $H_c(f)$, the filtering within the channel and $H_r(f)$, the receiving/equalizing filter.

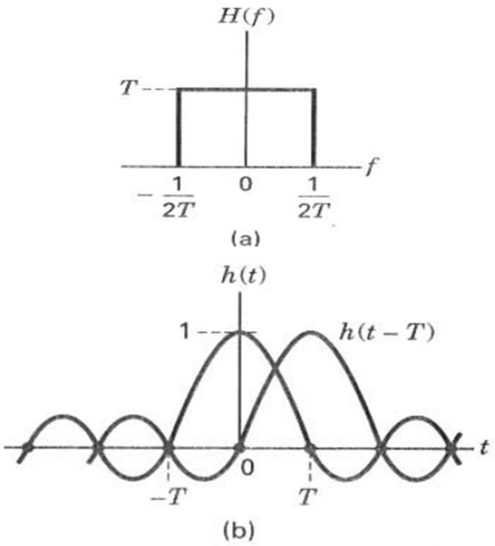
$$H(f) = H_t(f)\, H_c(f)\, H_r(f) \quad \text{…………(13)}$$

- The characteristic H(f) then represents the composite system transfer function due to all filtering at various locations throughout the transmitter/receiver chain.
- For a PCM waveform, the detector makes a symbol decision by comparing a sample of the received pulse to a threshold.
- The detector decides that a binary one was sent if the received pulse is positive and that a binary zero was sent, if the received pulse is negative.

- Due to the effects of system filtering, the received pulses can overlap one another as shown.
- The tail of a pulse can “smear” into adjacent symbol intervals, thereby interfering with the detection process and degrading the error performance.
- Such interference is termed as Inter Symbol Interference (ISI).
- Even in the absence of noise, the effects of filtering and channel induced distortion lead to ISI.
- Sometimes $H_c(f)$ is specified, and the problem remains to determine $H_t(f)$and $H_r(f)$, such that ISI is minimized at the output of $H_r(f)$.

Nyquist Criteria for zero ISI

- Nyquist investigated the problem of specifying a received pulse shape so that no ISI occurs at the detector.
- He showed that the theoretical maximum system bandwidth needed in order to detect R_s symbols/s, without ISI is $R_s/2$ Hz.
- This occurs when the System Transfer Function H(f) is made Rectangular, as shown.
- For baseband systems, when H(f) is such a filter with single-sided bandwidth 1/2T (the ideal Nyquist filter), its impulse response, the inverse Fourier transform of H(f) is of the form h(t) = sinc (t/T), as shown.
- This sinc (t/T) shaped pulse is called the ideal Nyquist pulse.



Nyquist channels for zero ISI.
(a) Rectangular system transfer function.
(b) Received pulse shape h(t) = sinc (t/T)

- The multiple lobes comprise a main lobe and side lobes called pre- and post-main lobe tails that are infinitely long.
- Nyquist established that if each pulse of a received sequence is of the form sinc (t/T), the pulses can be detected without ISI.
- As shown in figure, ISI can be avoided.
- There are two successive pulses, h(t) and h(t-T).
- Even though h (t) has long tails, the figure shows a tail passing through zero amplitude at the instant (t=T) when h (t-T) is to be sampled, and likewise all tails pass through zero amplitude when any other pulse of the sequence h(t-kT), k=∓1, ∓2, … is to be sampled.

- Therefore, assuming that the sampling timing is perfect, there will be no ISI degradation introduced.
- For baseband systems, the bandwidth required to detect 1/T such pulses (symbols) per second is equal to 1/2T.
- So, a system with bandwidth W= 1/2T = R_s/2 Hz, can support a maximum transmission rate of 2W = 1/T= R_s symbols/s (Nyquist bandwidth constraint) without ISI.

Ideal solution

- Thus, for ideal Nyquist filtering and zero ISI, the maximum possible symbol transmission rate per Hz, called the Symbol Rate Packing, is 2 symbols/Hz.
- It is to be noted that from the rectangular shaped transfer function of the ideal Nyquist filter and the infinite length of its corresponding pulse, that such ideal filters are not exactly realizable , they can only be approximately realized in practice.

- Nyquist filter and Nyquist pulse are used to describe the general class of filtering and pulse shaping that satisfy zero ISI at the sampling points.
- A Nyquist filter is one whose frequency transfer function can be represented by a sinc (t/T) function multiplied by another time function.
- Hence, there are countless number of Nyquist filters and corresponding pulse shaped.
- Amongst the class of Nyquist filters, the most popular ones are Raised Cosine filter and the Root Raised Cosine filter.

- A fundamental parameter for communication systems is Bandwidth Efficiency, R/W, whose units are bits/s/Hz.
- R/W, represents a measure of Data Throughput per Hz of Bandwidth and thus measures how efficiently any signalling technique utilizes the bandwidth resource.
- Since the Nyquist bandwidth constraint dictates that the theoretical maximum symbol rate packing without ISI is 2 symbols/s/Hz, what it says about the maximum number of bits/s/Hz? →
- It says nothing about bits directly.
- The constraints deals only with pulses or symbols, and the ability to detect their amplitude values without distortion from other pulses.
- To find R/W for any signalling scheme, one needs to know how many bits each symbol represents.

- Consider an M-ary PAM signalling set.
- Each symbol (comprising k bits) is represented by one of the M-pulse amplitudes.
- For k=6 bits per symbol, the symbol set size is $M=2^k$ = 64 amplitudes.
- Thus with 64-ary PAM, the theoretical maximum bandwidth efficiency that is possible without ISI is 12 bits/s/Hz.

Pulse Shaping to reduce ISI

- The more compact the signalling spectrum, the higher is the allowable data rate, or the greater is the number of users that can simultaneously be served.
- This has important implications to communication service providers, since greater utilization of the available bandwidth translates into greater revenue.
- For most communication systems, except spread-spectrum systems, the goal is to reduce the required system bandwidth as much as possible.
- Nyquist has provided the basic limitation to such bandwidth reduction.
- What happened if a system is forced to operate at smaller bandwidths than the constraint dictates? →
- The pulses would become spread in time which would degrade the system's error performance due to increased ISI.

- A judicious goal is to compress the bandwidth of the data impulses to some reasonably small bandwidth greater than the Nyquist minimum.
- This is done by Pulse Shaping with a Nyquist filter.
- If the band edge of the filter is steep, approaching the rectangle, then the signalling spectrum can be made most compact.
- But, such a filter has an impulse response duration approaching infinity.
- Each pulse extends into every pulse in the entire sequence.
- Long-time responses exhibit large amplitude tails nearest the main lobe of each pulse.
- Such tails are undesirable as they contribute zero ISI only when the sampling is performed at exactly the correct sampling time.
- When tails are large, small timing errors will result in ISI.
- Therefore, although a compact spectrum provides optimum bandwidth utilization, it is very susceptible to ISI degradation induced by timing errors.

The Raised Cosine Filter

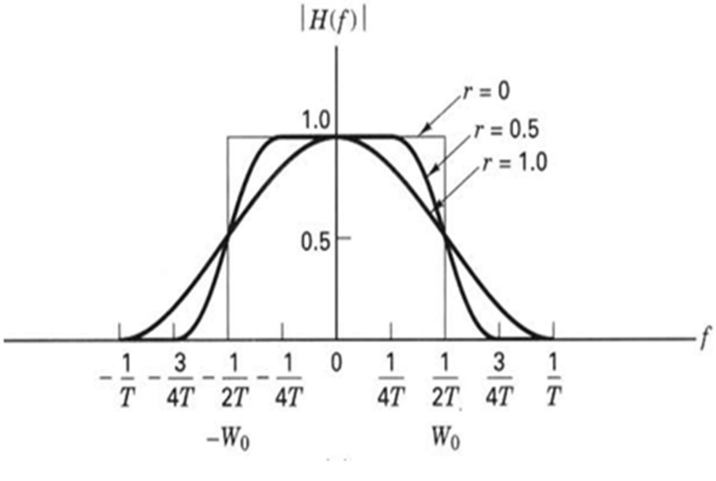
- The receiving Filter is often referred to as an Equalizing Filter as it is configured to compensate for the distortion caused by both the transmitter and the channel.
- The configuration of this Filter is chosen so as to optimize the composite system frequency transfer function H(f).
- One frequently used H(f) transfer function belonging to the Nyquist class (zero ISI at the sampling times) is called the Raised Cosine filter.

- It can be expressed as,

$$H(f) = \left\{ \begin{array}{ll} 1 & \text{for } |f| < 2W_0 - W \\ \cos^2\left(\dfrac{\pi}{4}\dfrac{|f| + W - 2W_0}{W - W_0}\right) & \text{for } 2W_0 - W < |f| < W \\ 0 & \text{for } |f| > W \end{array} \right\} \qquad \text{…………(14)}$$

- Where W is the absolute bandwidth and $W_0 = 1/2T$ represent the minimum Nyquist bandwidth for the rectangular spectrum and the -6 dB bandwidth (or half-amplitude point) for the Raised Cosine spectrum.

- The difference (W-W_0) is termed the Excess Bandwidth, which means additional bandwidth beyond the Nyquist minimum (i.e., for the rectangular spectrum, W is equal to W_0).
- The Roll-Off Factor is defined as $r=(W- W_0)/ W_0$, where $0 \leq r \leq 1$.
- It represents the Excess Bandwidth divided by the filter -6 dB Bandwidth (i.e., the fractional excess bandwidth).
- For a given W_0, the Roll-Off r specifies the required excess bandwidth as a fraction of W_0 and characterizes the steepness of the filter roll-off.

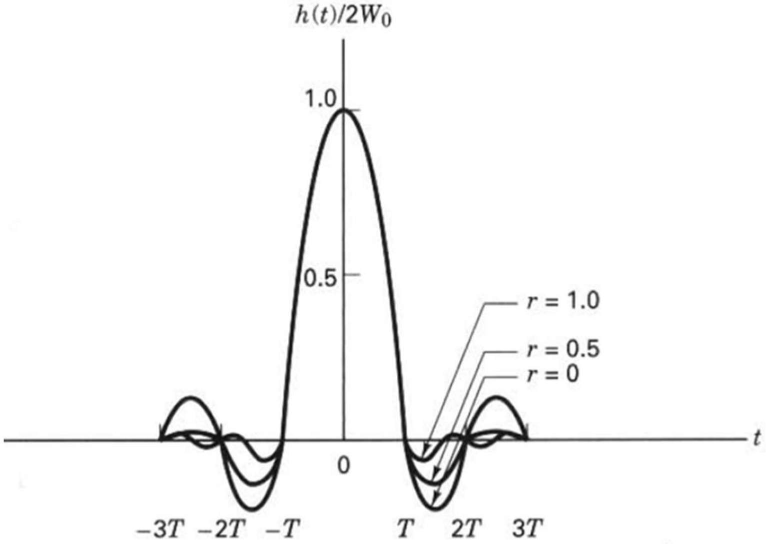


Raised-Cosine Filter characteristics - System Transfer Function

- The raised cosine characteristic is shown for a roll-off values of *r = 0*, *r = 0.5*, and *r = 1*.
- The *r = 0* roll-off is the Nyquist minimum bandwidth case.
- Note that when *r = 1*, the required excess bandwidth is 100%, and the tails are quite small.
- A system with such an overall spectral characteristic can provide a symbol rate of R_s symbols/s using a bandwidth of R_s Hz (twice the Nyquist minimum bandwidth), thus yielding a symbol rate packing of 1 symbol/s/Hz.

- The corresponding impulse response for the filter is,

$$h(t) = 2W_0(\text{sinc}\, 2W_0 t)\frac{\cos\,[2\pi(W - W_0)t]}{1 - [4(W - W_0)t\,]^2} \quad \text{…………(15)}$$



Raised-Cosine Filter characteristics -System Impulse Response

- It is plotted as shown in the diagram for *r = 0*, *r = 0.5*, and *r = 1*.
- The tails have zero value at each pulse sampling time, regardless of the roll-off value.
- Only an approximate implementation of a filter described by eqn(14) and a pulse shape described by eqn(15) possible, since the Raised Cosine spectrum is not physically realizable (for the same reason that the ideal Nyquist filter is not realizable).
- A realizable filter must have an impulse response of finite duration and exhibit a zero output prior to the pulse turn-on time.

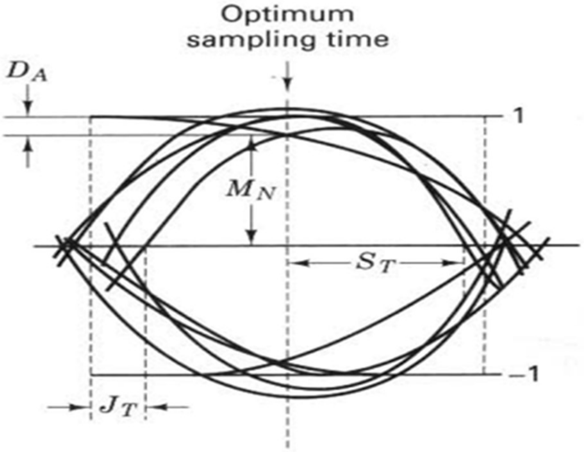
- A Pulse Shaping filter should satisfy <u>two requirements</u>.
- It should provide the <u>desired roll-off</u>, and it should be <u>realizable</u>, i.e., the impulse response need to be truncated to a finite length.
- Start with the <u>Nyquist bandwidth constrain</u>t that the minimum required system bandwidth W for a symbol rate of R_s symbols/s without ISI is $R_s/2$ Hz.
- A more general relationship between required bandwidth and symbol transmission rate involves the filter roll-off factor *r* and can be stated as,

$$W = \frac{1}{2}(1 + r)R_s \qquad \text{………… (16)}$$

- So, with $r = 0$, this equation describes the <u>minimum required bandwidth</u> for ideal Nyquist filtering.
- For $r > 0$, there is a <u>bandwidth expansion</u> beyond the Nyquist minimum.
- So, for this case, R_s is now less than twice the bandwidth.
- If the demodulator outputs one sample per symbol, then the Nyquist sampling theorem has been violated, since there are too few samples left with to reconstruct the analog waveform, with presence of aliasing.
- Since the family of raised cosine filters is characterized by zero ISI at the times that the symbols are sampled, <u>unambiguous detection</u> can still be achieved.

Eye Pattern

- An Eye Pattern is the display that results from measuring a system's response to baseband signals in a prescribed way.
- On the vertical plates of an oscilloscope, connect the receiver's response to a random pulse sequence.
- On the horizontal plates, connect a saw tooth wave at the signaling frequency.
- The horizontal time base of the oscilloscope is set equal to the symbol (pulse) duration.
- This setup superimposes the waveform in each signaling interval into a family of traces in a single interval (0, T).



Eye Pattern

- The Eye Pattern that results for bipolar pulse signaling is shown.
- Since the symbols stem from a random source, they are sometimes positive and sometimes negative, and the persistence of the CRT display allowed one to see the resulting pattern shaped as an eye.
- The width of the opening indicates the time over which sampling for detection might be performed.
- It is to be noted that, the optimum sampling time corresponds to the maximum eye opening, yielding the greatest protection against noise.
- If there were no filtering in the system, that is, if the bandwidth corresponding to the transmission of these data pulses were infinite, then the system response would yield rectangular pulse shapes.

- In that case, the pattern would look like a box rather than the eye.
- The range of amplitude differences labelled D_A is a measure of distortion caused by ISI, and the range of time differences of the zero crossings labelled J_T is a measure of the timing jitter.
- Measures of noise margin M_N and sensitivity-to-timing error S_T are also shown in the diagram.
- In general, the most frequent use of the eye pattern is for qualitatively assessing the extend of the ISI.
- As the eye closes, ISI is increasing.
- As the eye opens, ISI is decreasing.

Correlative Level Coding

Module-II Part-2

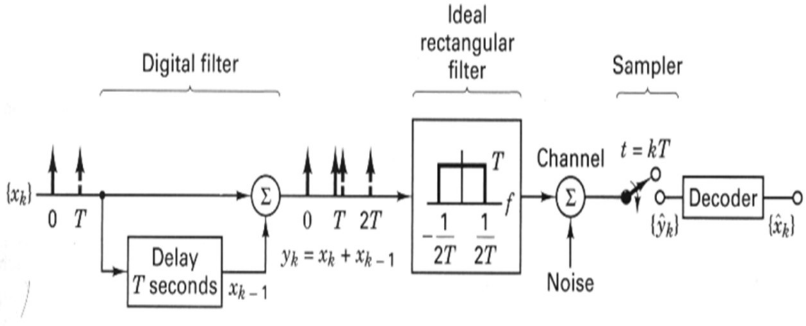
EC302 Digital Communication

Introduction

- In 1963, Adam Lender showed that it is possible to transmit 2W symbols/s with zero ISI, using the theoretical minimum bandwidth of W Hz, without infinite sharp filters, using the technique of Duobinary Signaling which is also known as Correlative Coding and Partial Response Signaling.
- The basic idea behind the Duobinary technique is to introduce some controlled amount of ISI into the datastream rather than trying to eliminate it completely.
- By introducing Correlated Interference between the pulses, and by changing the detection procedure, Lender was effectively able to cancel out the interference at the detector and thereby achieve the ideal symbol-rate packing of 2 symbols/s/Hz, an amount that had been considered unrealizable.

Duobinary Signaling

- Duobinary signaling introduces controlled ISI.
- Consider the model of the Duobinary coding process.



Duobinary Signaling

- Assume that a sequence of binary symbols $\{x_k\}$ is to be transmitted at the rate of R symbols/s over a system having an ideal rectangular spectrum of bandwidth W=R/2 = 1/2T Hz.
- Initially, the pulses pass through a simple digital filter, which incorporates a one digit delay to each incoming pulse, the filter adds the current value to the previous pulse.
- So, for every pulse to the input of the digital filter, there is summation of two pulses out.
- Each pulse of the sequence $\{y_k\}$ out of the digital filter can be expresses as,

$$y_k = x_k + x_{k-1} \qquad \text{……………(1)}$$

- Hence, the $\{y_k\}$ amplitudes are not independent as each y_k digit carries with it the memory of the prior digit.
- The ISI introduced to each y_k digit comes only from the preceding x_{k-1} digit.
- This correlation between the pulse amplitudes of $\{y_k\}$ can be thought of as the controlled ISI introduced by the Duobinary coding.
- Controlled interference is the essence of this novel technique, since at the detector, such controlled interference can be as easily removed as it was added.

- The sequence $\{y_k\}$ is followed by the ideal Nyquist filter that does not introduce any ISI.
- At the receiver sampler, the sequence $\{y_k\}$ can be recovered exactly in the absence of noise.
- Since all system experience noise contamination, refer to the received $\{y_k\}$ as the estimate of $\{y_k\}$ and denote it $\{\hat{y}_k\}$.
- Removing the controlled interference with the Duobinary decoder yields an estimate of $\{x_k\}$ which is denoted as $\{\hat{x}_k\}$.

Duobinary Decoding

- If the binary digit x_k is equal to ±1, then by using eqn(1), y_k has one of the three possible values: +2, 0, or -2.
- The Duobinary code results in a <u>three-level output</u>.
- In general for M-ary transmission, partial response signalling results in 2M-1 output levels.
- The decoding procedure involved the <u>inverse</u> of the coding procedure, namely, subtracting the x_k-1 decision from the y_k digit.

Example -1.

<u>Duobinary coding and decoding</u>

Demonstrate Duobinary coding and decoding for the following sequence $\{x_k\}$ = 0 0 1 0 1 1 0. Let the first bit be a start-up digit, not part of the data.

<u>Sol:</u>

Binary digit sequence $\{x_k\}$:	0	0	1	0	1	1	0
Bipolar amplitudes $\{x_k\}$:	-1	-1	+1	-1	+1	+1	-1
Coding rule $y_k = x_k + x_{k-1}$:		-2	0	0	0	2	0

Decoding decision rule:

If $\hat{y}_k$ = 2, decide that $\hat{x}_k$= +1 (or binary one).

If $\hat{y}_k$ = -2, decide that $\hat{x}_k$= -1 (or binary one).

If $\hat{y}_k$ = 0, decide opposite of the previous decision.

Decoded bipolar sequence $\{\hat{x}_k\}$:	-1	+1	-1	+1	+1	-1
Decoded binary sequence $\{\hat{x}_k\}$:	0	1	0	1	1	0

Points to Ponder…

- The decision rule simply implements the subtraction of each $\hat{x}_{k-1}$ decision from each $\hat{y}_{k.}$
- One drawback of this decision technique is that once an error is made, it tends to propagate, causing further errors, since present decisions depend on prior decisions.
- A means of avoiding this error propagation is known as Precoding.

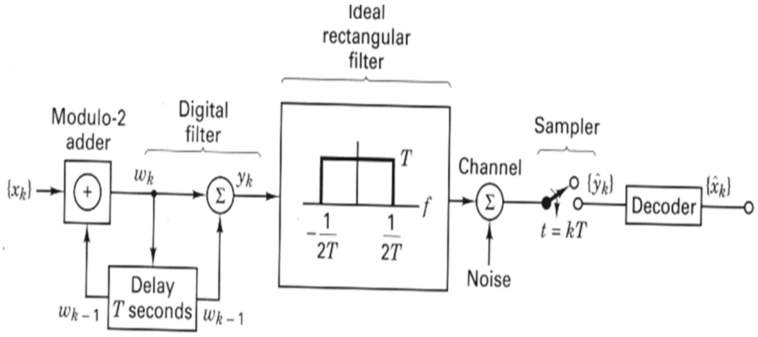
Precoding

- Precoding is accomplished by first differentially encoding the $\{x_k\}$ binary sequence into a new $\{w_k\}$ binary sequence by means of the equation:

$$w_k = x_k \oplus w_{k-1} \quad \text{..........} \ (2)$$

- The $\{w_k\}$ binary sequence is then converted to a bipolar pulse sequence, and the coding operation proceeds in the say way as before.
- However, with Precoding, the detection process is quite different from the detection of ordinary Duobinary coding.

The Precoding model is shown below in the diagram, whereby it is clear that the modulo-2 addition producing the precoded $\{w_k\}$ sequence is performed on the binary digits, while the digital filtering producing the $\{y_k\}$ sequence is performed on the bipolar pulses.



Precoded Duobinary Signalling

Example:2 Duobinary Precoding

Demonstrate Duobinary coding and decoding when using Precoding for the following sequence {x_k} = 0 0 1 0 1 1 0.

Sol:

Binary digit sequence {x_k}:	0	0	1	0	1	1	0
Precoded Sequence: $w_k = x_k \oplus w_{k-1}$	0	0	1	1	0	1	1
Bipolar amplitudes {x_k}: $y_k = w_k + w_{k-1}$:	-1	-1	+1	+1	-1	+1	+1
Coding rule		-2	0	+2	0	0	+2

- Decoding decision rule:
  If $\hat{y}_k = \pm 2$, decide that $\hat{x}_k$= binary zero.
  If $\hat{y}_k$ = 0, decide that $\hat{x}_k$= binary one.

- Decoded binary sequence {$\hat{x}_k$}:

  0 1 0 1 1 0

- The differential Precoding enables to decode the {$\hat{y}_k$} sequence by making a decision on each received sample singly, without resorting to prior decisions that could be in error.
- The major advantage is that in the event of a digit error to noise, such an error does not propagate to other digits.
- The first bit in the differentially precoded binary sequence {w_k}is an arbitrary choice.
- If the start-up bit in {w_k} had been chosen to be a binary one instead of binary zero, the decoded result would have been the same.

Generalized Partial response signalling.

- The duobinary transfer function has a digital filter that incorporates a one-digit delay followed by an ideal rectangular transfer function.
- Examine an equivalent model.
- The Fourier transform of a delay can be given as $e^{-j2\pi fT}$.
- So the input digital filter can be characterized as the frequency transfer function,

$$H_1(f) = 1 + e^{-j2\pi fT} \qquad \text{……….. (3)}$$

- The transfer function of the ideal rectangular filter is,

$$H_2(f) = \begin{cases} T & \text{for } |f| < \frac{1}{2T} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{……….. (4)}$$

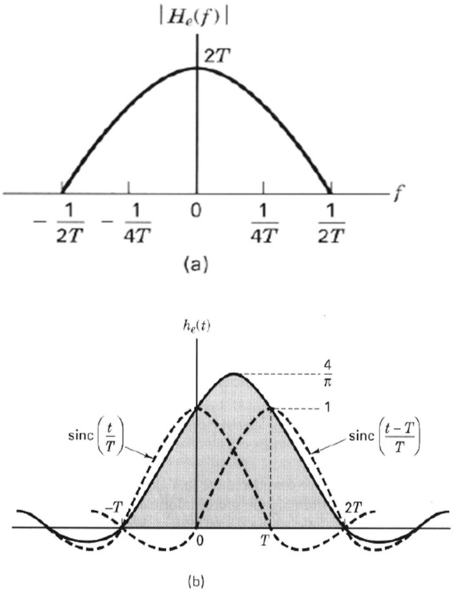
- The overall equivalent transfer function of the digital filter cascaded with the ideal rectangular filter is then given by,

$$\begin{aligned} H_e(f) &= H_1(f)H_2(f) \qquad \text{for } |f| < \frac{1}{2T} \\ &= (1 + e^{-j2\pi fT})T \qquad \text{……….. (5)} \\ &= T(e^{j\pi fT} + e^{-j\pi fT})e^{-j\pi fT} \end{aligned}$$

$$|H_e(f)| = \begin{cases} 2T\cos \pi fT & \text{for } |f| < \frac{1}{2T} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{……….. (6)}$$

- Thus the composite transfer function for the cascaded digital and rectangular filters, has a gradual roll-off to the band edge as shown below.
- The transfer function can be approximated by using realizable analog filtering, a separate digital filter is not needed.
- The duobinary equivalent $H_e(f)$ is called a Cosine Filter.
- The Cosine Filter is not the same as the Raised Cosine Filter.
- The corresponding impulse response $h_e(t)$ is got by taking the inverse Fourier transform of $H_e(f)$ and is given as,

$$h_e(t) = \text{sinc}\left(\frac{t}{T}\right) + \text{sinc}\left(\frac{t-T}{T}\right) \quad \text{………. (7)}$$



Duobinary Transfer Function and Pulse Shape
(a) Cosine Filter (b) Impulse response of the Cosine Filter

- For every impulse $\delta(t)$ at the input of the model, the output is $h_e(t)$ with an appropriate polarity.
- Notice that there are only two nonzero samples at T-second intervals, giving rise to controlled ISI from the adjacent bit.
- The introduced ISI is eliminated by use of the decoding procedure.
- Although the Cosine Filter is noncausal and therefore nonrealizable, it can easily be approximated.
- The implementation of the precoded duobinary technique can be accomplished by first differentially encoding the binary sequence $\{x_k\}$ into the sequence $\{w_k\}$.
- The pulse sequence $\{w_k\}$ is than filtered by the equivalent cosine characteristic.

Comparison of Binary with Duobinary Signalling

- The duobinary technique introduces correlation between pulse amplitudes, whereas the more restrictive Nyquist criterion assumes that the transmitted pulse amplitudes are independent of one another.
- Duobinary signalling exploits introduced correlation to achieve zero ISI signal transmission using a smaller system bandwidth than is otherwise possible.
- But there is trade-off involved.
- Duobinary coding required three levels, compared with the usual tow levels of binary coding.
- For a fixed amount of signal power, the ease of making reliable decisions is inversely related to the number of levels that must be distinguished in each waveform.

- Therefore, although duobinary signalling achieves the zero ISI requirements with minimum bandwidth, duobinary signalling also required more power than binary signalling, for equivalent performance against noise.
- For a given probability of bit error (P_B), duobinary signalling required approximately 2.5 dB greater SNR than binary signalling, while using only 1/(1+r) the bandwidth that binary signalling requires, where r is the filter roll-off.

Signal Space Analysis:
Module – III Part-1

EC302 Digital Communication

Introduction

The steady transition from analog communications to digital communications has been empowered by several factors:

1. Ever-increasing advancement of digital silicon chips, digital signal processing, and computers, which, in turn, has prompted further enhancement in digital silicon chips, thereby repeating the cycle of improvement.

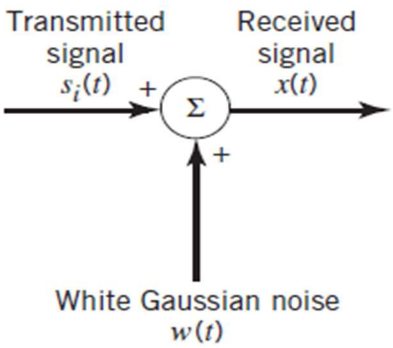
2. Improved reliability, which is afforded by digital communications to a much greater extent than is possible with analog communications.

3. Broadened range of multiplexing of users, which is enabled by the use of digital modulation techniques.

4. Communication networks, for which, in one form or another, the use of digital communications is the preferred choice.

- As an example, consider the remote connection of two digital computers, with one computer acting as the information source by calculating digital outputs based on observations and inputs fed into it.
- The other computer acts as the recipient of the information.
- The source output consists of a sequence of 1s and 0s, with each binary symbol being emitted every T_b seconds.
- The transmitting part of the digital communication system takes the 1s and 0s emitted by the source computer and encodes them into distinct signals denoted by $s_1(t)$ and $s_2(t)$, respectively, which are suitable for transmission over the analog channel.
- Both $s_1(t)$ and $s_2(t)$ are real-valued energy signals, as shown by,

$$E_i = \int_0^{T_b} s_i^2(t)\, dt, \qquad i = 1, 2 \qquad \text{.............. (1)}$$



AWGN model of a channel.

- With the analog channel represented by an AWGN model, the received signal is defined by,

$$\therefore \quad x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T_b \\ i = 1, 2 \end{cases} \quad \text{..............} (2)$$

- where w(t) is the channel noise.
- The receiver has the task of observing the received signal x(t) for a duration of T_b seconds and then making an estimate of the transmitted signal $s_i(t)$ or equivalently the ith symbol, i = 1, 2.
- But, due to channel noise, the receiver will make occasional errors.
- The requirement, therefore, is to design the receiver so as to minimize the average Probability of Symbol Error.

To minimize the average probability of symbol error between the receiver output and the symbol emitted by the source, make the digital communication system as reliable as possible.

To achieve this important design objective in a generic setting that involves an M-ary alphabet whose symbols are denoted by m_1, m_1, …,m_M , there are two basic issues:

1. To optimize the design of the Receiver so as to minimize the average probability of symbol error.

2. To choose the set of signals $s_1(t)$, $s_2(t)$,…, $s_M(t)$ for representing the symbols m_1, m_1, …,m_M, respectively, since this choice affects the average probability of symbol error.

Geometric Representation of Signals

- The essence of geometric representation of signals is to represent any set of M energy signals $\{s_i(t)\}$ as linear combinations of N orthonormal basis functions, where $N \leq M$.
- Given a set of real-valued energy signals, $s_1(t)$, $s_2(t), \ldots, s_M(t)$ each of duration T seconds,

$$\therefore \quad s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{cases} \qquad \ldots\ldots\ldots\ldots\ (3)$$

- where the coefficients of the expansion are defined by,

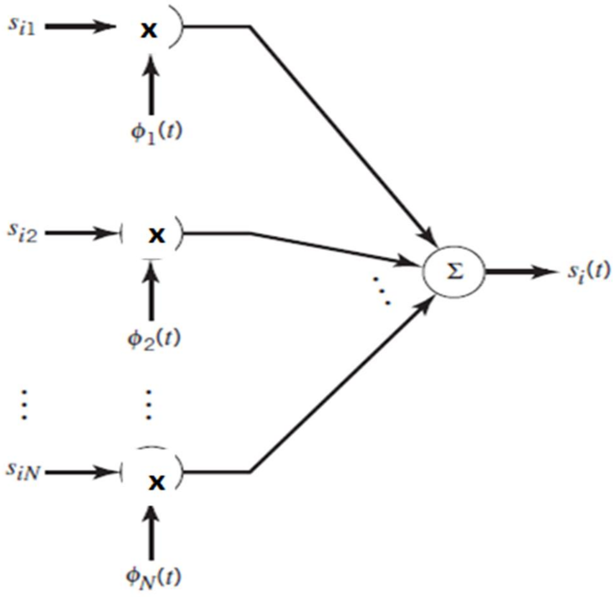
$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,\mathrm{d}t, \qquad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases} \qquad \ldots\ldots\ldots\ldots\ (4)$$

- The real-valued basis functions $\phi_1(t)$, ϕ_2 (t), …, $\phi_N(t)$ form an orthonormal set,

$$\therefore \quad \int_0^T \phi_i(t)\phi_j(t)\,\mathrm{d}t = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \ldots\ldots\ldots\ldots\ (5)$$

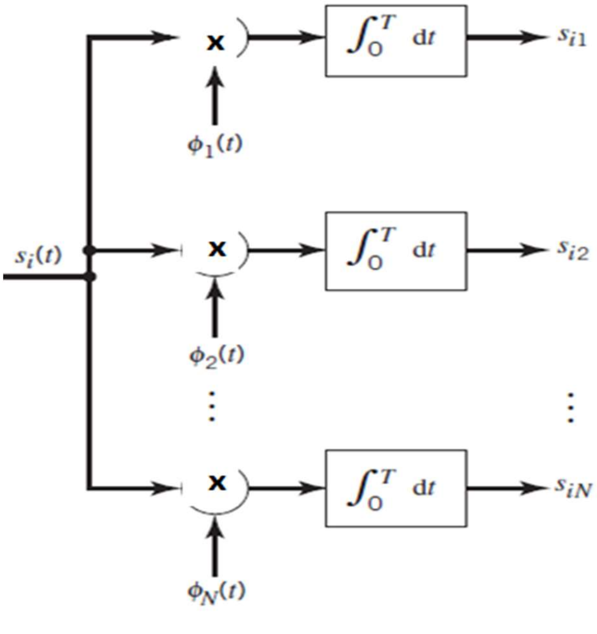
- where δ_{ij} is the Kronecker delta.
- The first condition of (5) states that each basis function is normalized to have unit energy.
- The second condition states that the basis functions $\phi_1(t)$, ϕ_2 (t), …, $\phi_N(t)$ are orthogonal w.r.t each other over the interval $0 \leq t \leq T$.

- For prescribed i, the set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional signal vector, denoted by vector $\mathbf{s}_i$.
- The important point to note here is that the vector $\mathbf{s}_i$ bears a one-to-one relationship with the transmitted signal $s_i(t)$.
- Given the N elements of the vector $\mathbf{s}_i$ operating as input, use the scheme shown next to generate the signal $s_i(t)$, which follows directly from eqn(3).
- This figure consists of a bank of N multipliers with each multiplier having its own basis function followed by a summer.
- This scheme is considered as a Synthesizer.



(a) Synthesizer for generating the signal $s_i(t)$.

- Conversely, given signals $s_i(t)$, i = 1, 2, …, M, operating as input, use the scheme shown below to calculate the coefficients s_{i1}, s_{i2}, …, s_{iN} which follows directly from (4).
- This scheme consists of a bank of N product integrators or correlators with a common input, and with each one of them supplied with its own basis function.
- The scheme is considered as an Analyzer.



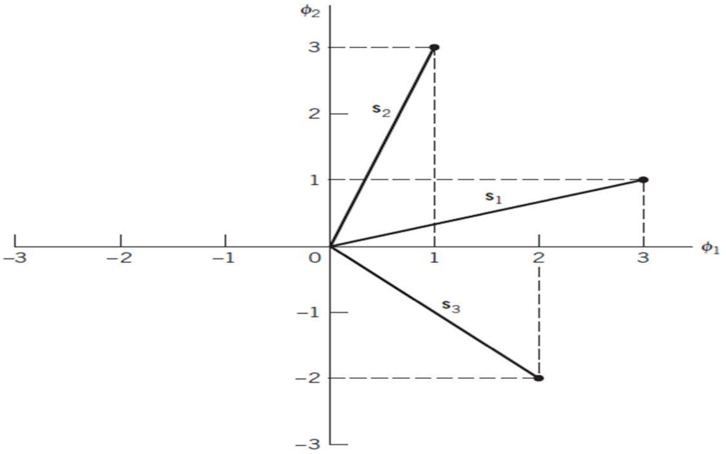
Analyzer for reconstructing the signal vector $\{s_i\}$.

- Each signal in the set $\{s_i(t)\}$ is completely determined by the signal vector,

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \qquad i = 1, 2, \ldots, M \qquad \text{.............. (6)}$$

- The notion of two- and three dimensional Euclidean spaces can be extended to an N-dimensional Euclidean space, with the set of signal vectors $\{\mathbf{s}_i | i = 1, 2, \ldots, M\}$ as defining a corresponding set of M points in an N-dimensional Euclidean space, with N mutually perpendicular axes labelled $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$.
- This N-dimensional Euclidean space is called the Signal Space.

- This form of representation is shown below for the case of a two-dimensional signal space with three signals; that is, N = 2 and M = 3.



Geometric representation of signals for the case when N = 2 and M = 3.

- In an N-dimensional Euclidean space, define lengths of vectors and angles between vectors.
- Denote the length (also called the absolute value or norm) of a signal vector $\mathbf{s}_i$ by the symbol $\|\mathbf{s_i}\|$.
- The squared length of any signal vector $\mathbf{s}_i$ is defined to be the inner product or dot product of $\mathbf{s}_i$ with itself, as given by,

$$\|\mathbf{s}_i\|^2 = \mathbf{s}_i^{\mathbf{T}}\mathbf{s}_i = \sum_{j=1}^{N} s_{ij}^2, \qquad i = 1, 2, \ldots, M \qquad \text{.............. (7)}$$

- where s_{ij} is the jth element of $\mathbf{s}_i$ and the superscript T denotes matrix transposition.

- There is an interesting relationship between the energy content of a signal and its representation as a vector.
- By definition, the energy of a signal $s_i(t)$ of duration T seconds is,

$$E_i = \int_0^T s_i^2(t)\,\mathrm{d}t, \qquad i = 1, 2, \ldots, M \qquad \text{.............. (8)}$$

- Therefore, substituting eqn(3) into eqn(8),

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\phi_k(t)\right]\mathrm{d}t$$

- Interchanging the order of summation and integration, and then rearranging terms,

$$E_i = \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}s_{ik}\int_0^T \phi_j(t)\,\phi_k(t)\,\mathrm{d}t \qquad \text{.............. (9)}$$

- Since, by definition, the $\phi_j(t)$ form an orthonormal set in accordance with the two conditions of eqn(5), eqn(9) reduces simply to,

$$E_i = \sum_{j=1}^{N} s_{ij}^2 = \|\mathbf{s}_i\|^2 \quad \text{.............. (10)}$$

- Thus, (7) and (10) show that the energy of an energy signal $s_i(t)$ is equal to the squared length of the corresponding signal vector $\mathbf{s}_i(t)$.
- In the case of a pair of signals $s_i(t)$ and $s_k(t)$ represented by the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$, respectively,

$$\int_0^T s_i(t) s_k(t)\,\mathrm{d}t = \mathbf{s}_i^{\mathrm{T}} \mathbf{s}_k \quad \text{.............. (11)}$$

- The inner product of the energy signals $s_i(t)$ and $s_k(t)$ over the interval $[0,T]$ is equal to the inner product of their respective vector representations $\mathbf{s}_i$ and $\mathbf{s}_k$.
- Note that the inner product is invariant to the choice of basis functions $\{\phi_j(t)\}_{j=1}^{N}$, in that it only depends on the components of the signals $s_i(t)$ and $s_k(t)$ projected onto each of the basis functions.

- Yet another useful relation involving the vector representations of the energy signals $s_i(t)$ *and* $s_k(t)$ is described by,

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2$$

$$= \int_0^T (s_i(t) - s_k(t))^2 \, dt \qquad \text{.............. (12)}$$

- where $\|\mathbf{s}_i - \mathbf{s}_k\|^2$ is the Euclidean distance d_{ik} between the points represented by the signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$.

- To complete the geometric representation of energy signals, need to have a representation for the angle θ_{ik} subtended between two signal vectors $\mathbf{s}_i$ and $\mathbf{s}_k$.
- By definition, the cosine of the angle θ_{ik} is equal to the inner product of these two vectors divided by the product of their individual norms,

$$\cos(\theta_{ik}) = \frac{\mathbf{s}_i^{\mathrm{T}} \mathbf{s}_k}{\|\mathbf{s}_i\| \|\mathbf{s}_k\|} \qquad \text{.............. (13)}$$

- The two vectors $\mathbf{s}_i$ and $\mathbf{s}_k$ are thus orthogonal or perpendicular to each other if their inner product $\mathbf{s}_i^{\mathrm{T}}\mathbf{s}_k$ is zero, in which case $\theta_{ik} = 90°$.

Gram–Schmidt Orthogonalization Procedure

- The Gram–Schmidt Orthogonalization procedure uses a complete Orthonormal set of basis functions.
- To proceed with the formulation of this procedure, suppose there is a set of *M* energy signals denoted by $s_1(t), s_2(t), \ldots, s_M(t)$.
- Starting with $s_1(t)$ chosen from this set, the first basis function is defined by,

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \qquad \text{............. (17)}$$

- where E_1 is the energy of the signal $s_1(t)$.

$$s_1(t) = \sqrt{E_1}\phi_1(t) = s_{11}(t)\phi_1(t)$$

$$s_{11} = \sqrt{E_1}$$

- $\phi_1(t)$ has unit energy as required.
- Next, using the signal $s_2(t)$, define the coefficient s_{21} as,

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt$$

- Introduce a new intermediate function,

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \qquad \text{............. (18)}$$

- which is orthogonal to $\phi_1(t)$ over interval $0 \leq t \leq T$ by the definition of s_{21} and the fact that the basis function $\phi_1(t)$ has unit energy.

- Define the second basis function as,

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,\mathrm{d}t}} \quad \text{............. (19)}$$

- Substituting eqn(18) into (19) and simplifying,

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \quad \text{............. (20)}$$

- where E_2 is the energy of the signal $s_2(t)$.
- From eqn (19), it is seen that, $\int_0^T \phi_2^2(t)\,\mathrm{d}t = 1$
- in which case eqn (20) yields, $\int_0^T \phi_1(t)\phi_2(t)\,\mathrm{d}t = 0$
- Therefore, $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal pair as required.

- Continuing the procedure in this manner, in general, define,

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t) \quad \text{............. (21)}$$

- where the coefficients s_{ij} are themselves defined by,

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,\mathrm{d}t, \qquad j = 1, 2, \ldots, i-1$$

- For $i = 1$, the function $g_i(t)$ reduces to $s_i(t)$.
- Given the $g_i(t)$, define the set of basis functions,

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,\mathrm{d}t}}, \qquad j = 1, 2, \ldots, N \quad \text{............. (22)}$$

- which form an orthonormal set.

The dimension N is less than or equal to the number of given signals, M, depending on one of two possibilities:

1. The signals $s_1(t)$, $s_2(t)$, …, $s_M(t)$ form a linearly independent set, in which case $N = M$.
2. The signals $s_1(t)$, $s_2(t)$, …, $s_M(t)$ are not linearly independent, in which case $N < M$ and the intermediate function $g_i(t)$ is zero for $i > N$.

Points to Ponder

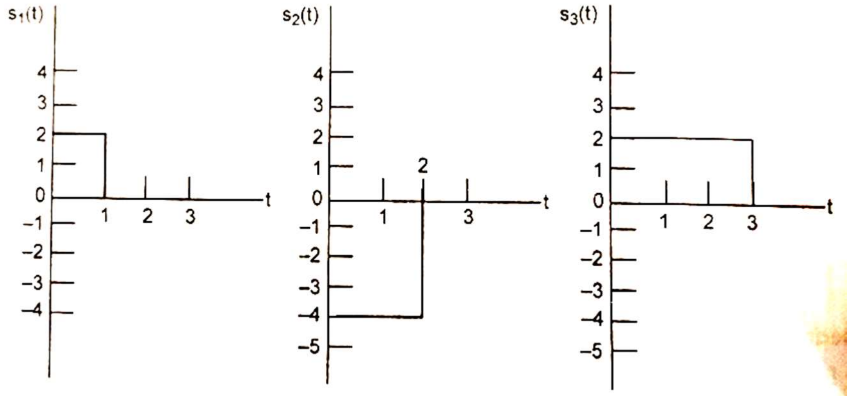
1. Unlike the Fourier series expansion of a periodic signal or the Sampled representation of a band-limited signal, the Gram–Schmidt Orthogonalization procedure is not restricted to be in terms of sinusoidal functions (as in the Fourier series) or sinc functions of time (as in the sampling process).

2. The expansion of the signal $s_i(t)$ in terms of a finite number of terms is not an approximation where only the first N terms are significant, instead it is an exact expression, where N and only N terms are significant.

Example -1

(a) Using the Gram-Schmidt orthogonalization procedure, find a set of orthonormal basis functions to represent the three signals $s_1(t)$, $s_2(t)$, and $s_3(t)$

(b) Express each of these signals in terms of the set of basis functions found in part (a).



Sol: (*a*) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

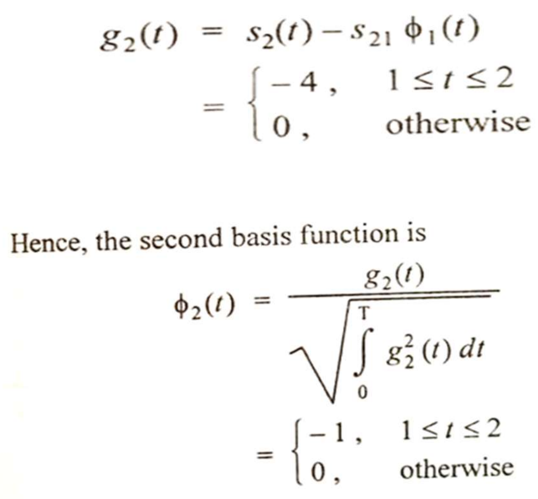
$$E_1 = \int_0^1 (2)^2\, dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\,\phi_1(t)\, dt = \int_0^T (-4)(1)\, dt = -4$$



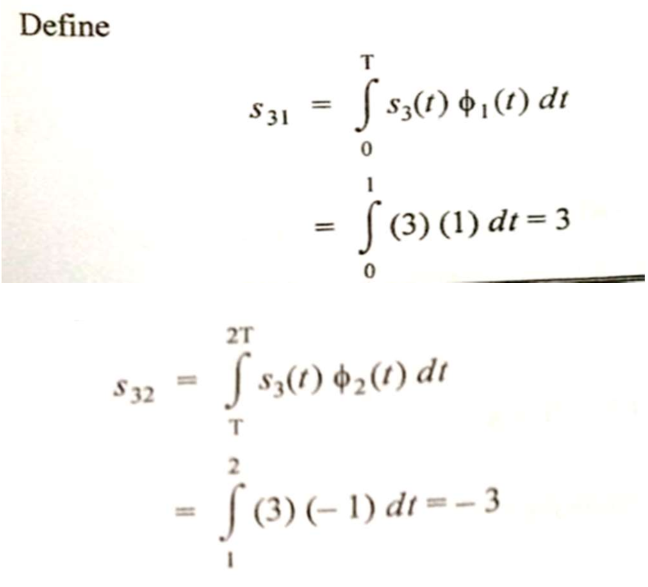
$$g_2(t) = s_2(t) - s_{21}\,\phi_1(t)$$

$$= \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int\limits_0^T g_2^2(t)\,dt}}$$

$$= \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



Define

$$s_{31} = \int\limits_0^T s_3(t)\,\phi_1(t)\,dt$$

$$= \int\limits_0^1 (3)\,(1)\,dt = 3$$

$$s_{32} = \int\limits_T^{2T} s_3(t)\,\phi_2(t)\,dt$$

$$= \int\limits_1^2 (3)\,(-1)\,dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\,\phi_1(t) - s_{32}\,\phi_2(t)$$

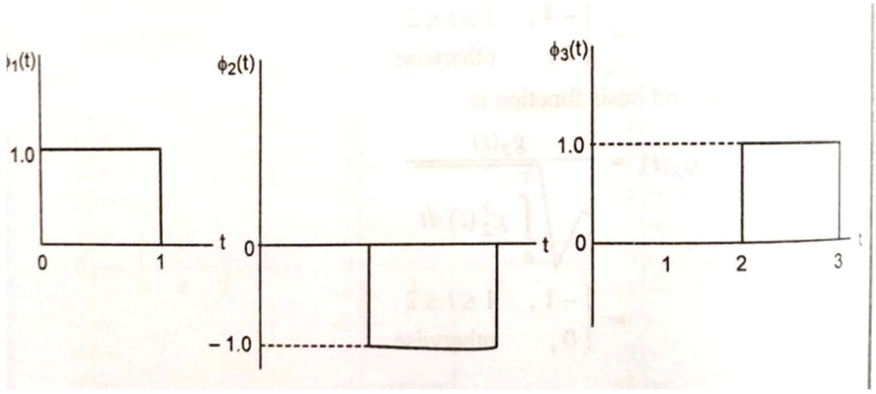
$$= \begin{cases} 3\,, & 2 \le t \le 3 \\ 0\,, & \text{otherwise} \end{cases}$$

Hence, the third basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int\limits_0^T g_3^2(t)\,dt}}$$

$$= \begin{cases} 1\,, & 2 \le t \le 3 \\ 0\,, & \text{otherwise} \end{cases}$$

The three basis functions are as follows (graphically)



(*b*)

$$s_1(t) = 2\,\phi_1(t)$$

$$s_2(t) = -4\,\phi_1(t) + 4\,\phi_2(t)$$

$$s_3(t) = 3\,\phi_1(t) - 3\,\phi_2(t) + 3\,\phi_3(t)$$

Transmission Over AWGN Channel:

Module – III Part-2

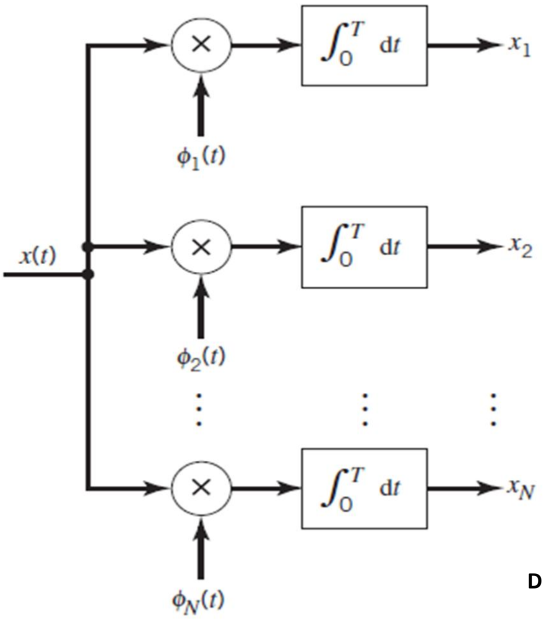
EC302 Digital Communication

Conversion of the Continuous AWGN Channel into a Vector Channel

- Suppose that the input to the bank of N product integrators or correlators is the received signal *x(t)* defined as:

$$x(t) = s_i(t) + w(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases} \qquad \ldots(23)$$

- *w(t)* is a sample function of a white Gaussian noise process W(t) of zero mean and power spectral density $N_0/2$.
- The output of correlator j is the sample value of a random variable X_j



Detector or Demodulator

The output of correlator j is the sample value of a random variable X_j

$$x_j = \int_0^T x(t)\phi_j(t)\,\mathrm{d}t$$

…(24)

$$= s_{ij} + w_j, \qquad j = 1, 2, \dots, N$$

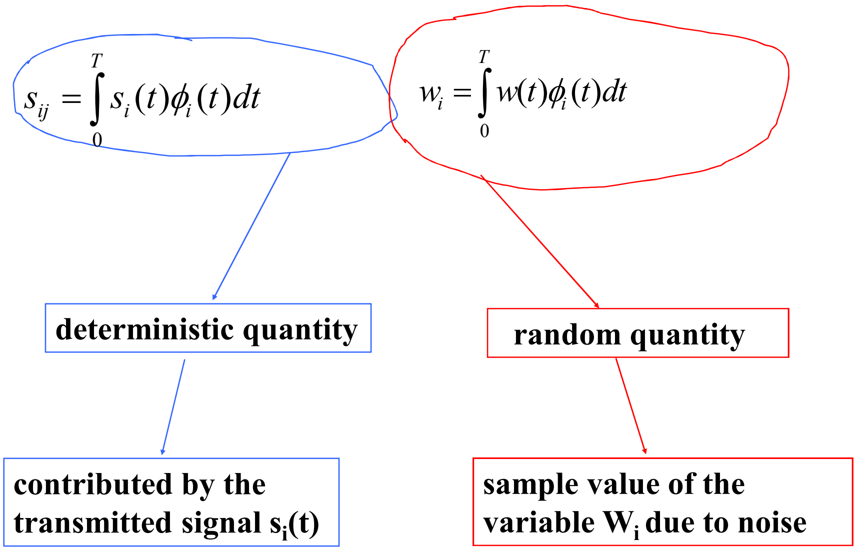
$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,\mathrm{d}t$$

Deterministic quantity!!

$$w_j = \int_0^T w(t)\phi_j(t)\,\mathrm{d}t$$

Random quantity!!

…..(25)



$$s_{ij} = \int_0^T s_i(t)\phi_i(t)dt$$

$$w_i = \int_0^T w(t)\phi_i(t)dt$$

deterministic quantity

contributed by the transmitted signal s_i(t)

random quantity

sample value of the variable W_i due to noise

- Consider a random process *X'(t),* with *x'(t),* a sample function which is related to the received signal *x(t)* as follows:

$$x'(t) = x(t) - \sum_{j=1}^{N} x_j\phi_j(t) \qquad \text{.....(26)}$$

$$x'(t) = x(t) - \sum_{j=1}^{N}(s_{ij} + w_j)\phi_j(t)$$

$$= w(t) - \sum_{j=1}^{N} w_j\phi_j(t)$$

$$= w'(t) \qquad \text{.....(27)}$$

- *which means that **the sample function x'(t) depends only on the channel noise!***

- The received signal can be expressed as:

$$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + x'(t)$$

$$= \sum_{j=1}^{N} x_j \phi_j(t) + w'(t) \qquad \text{.....(28)}$$

Statistical Characterization

- The received signal (output of the correlator) is a random signal.
- To describe it we need to use statistical methods – mean and variance. ??
- The assumptions are:
  - X(t) denotes a random process, a sample function of which is represented by the received signal x(t).
  - $X_j(t)$ denotes a random variable whose sample value is represented by the correlator output $x_j(t)$, j = 1, 2, …N.
  - We have assumed AWGN, so the noise is Gaussian, so X(t) is a Gaussian process and being a Gaussian RV, X_j is described fully by its Mean value and Variance.

Mean Value

- Let $W_{j,}$ denote a random variable, represented by its sample value w_j, produced by the j^{th} correlator in response to the Gaussian noise component w(t).
- So it has zero mean (by definition of the AWGN model)

$$\begin{aligned} \mu_{x_j} &= E\left[X_j\right] \\ &= E\left[s_{ij} + W_j\right] \\ &= s_{ij} + E[W_j] \\ \mu_{x_j} &= s_{ij} \end{aligned} \qquad \text{.....(29)}$$

- …then the ***mean of X_j*** depends only on s_{ij}

Variance

$$\begin{aligned} \sigma^2_{X_j} &= \text{var}[X_j] \\ &= \mathbb{E}[(X_j - s_{ij})^2] \\ &= \mathbb{E}[W_j^2] \end{aligned} \qquad \text{.....(29)}$$

$$W_j = \int_0^T W(t)\phi_j(t)\,\mathrm{d}t$$

$$\begin{aligned} \sigma^2_{X_j} &= \mathbb{E}\int_0^T W(t)\phi_j(t)\,\mathrm{d}t \int_0^T W(u)\phi_j(u)\,\mathrm{d}u \\ &= \mathbb{E}\left[\int_0^T\int_0^T \phi_j(t)\phi_j(u)W(t)W(u)\,\mathrm{d}t\,\mathrm{d}u\right] \end{aligned} \qquad \text{.....(30)}$$

- Interchanging the order of integration and expectation,

$$\sigma_{X_j}^2 = \int_0^T \int_0^T \phi_j(t)\phi_j(u)\mathbb{E}[W(t)W(u)]\, dt\, du$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)R_W(t, u)\, dt\, du \qquad (31)$$

- $R_w(t,u)$ → Autocorrelation function of the noise process

- Since the noise is stationary and with a constant power spectral density, it can be expressed as:

$$R_w(t,u) = \frac{N_0}{2}\delta(t-u) \qquad \text{.....(32)}$$

- After substitution for the variance we get:

$$\sigma_{X_j}^2 = \frac{N_0}{2}\int_0^T \int_0^T \phi_j(t)\phi_j(u)\delta(t-u)\, dt\, du$$
$$= \frac{N_0}{2}\int_0^T \phi_j^2(t)\, dt \qquad \text{.....(33)}$$

- And since $\phi_j(t)$ has unit energy for the ***variance*** we finally have:

$$\sigma_{X_j}^2 = \frac{N_0}{2}, \qquad \text{for all } j \qquad \text{.....(34)}$$

- ***Correlator outputs, denoted by X_j have variance equal to the power spectral density $N_0/2$ of the noise process W(t).***
- So, all the N correlator o/ps denoted by X_j with j=1,2,…N have a variance equal to PSD $N_0/2$ of W(t).
- ***Show that …***
- X_j are mutually uncorrelated (use Covariance of correlator o/ps)
- X_j are statistically independent (follows from above because X_j are Gaussian)

Define the vector

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$ **36**

x is called the observation vector.

The elements are indep. Gaussian RVs with means s_{ij}

and $N_0/2$ variances (for sample $s_i(t)$)

The conditional pdf given that $s_i(t)\,(m_i)$ was transmitted is

$$f_{\mathrm{x}}(\mathrm{x}|m_i) = \prod_{j=1}^{N} f_{x_j}(x_j|m_i) \ , \ i = 1,2,\ldots,M$$ 37

Any channel that satisfies 37 is called a $\boxed{\text{memoryless channel}}$

$$f_{x_j}(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_j - s_{ij})^2\right], \quad \begin{matrix} j = 1,2,\ldots,N \\ i = 1,2,..,M \end{matrix}$$ 38

$$f_{\mathrm{x}}(\mathrm{x}|m_i) = (\pi N_0)^{-\frac{N}{2}} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right] , i = 1,2,\ldots,M$$ 39

Recall $$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + w'(t)$$

$w'(t)$ is a zero - mean Gaussian process and indep. of $\{X_j\}$

$$E[X_j W'(t_k)] = 0 \ , \ \begin{cases} j = 1,2,\ldots,N \\ 0 \le t_k \le T \end{cases}$$ 40

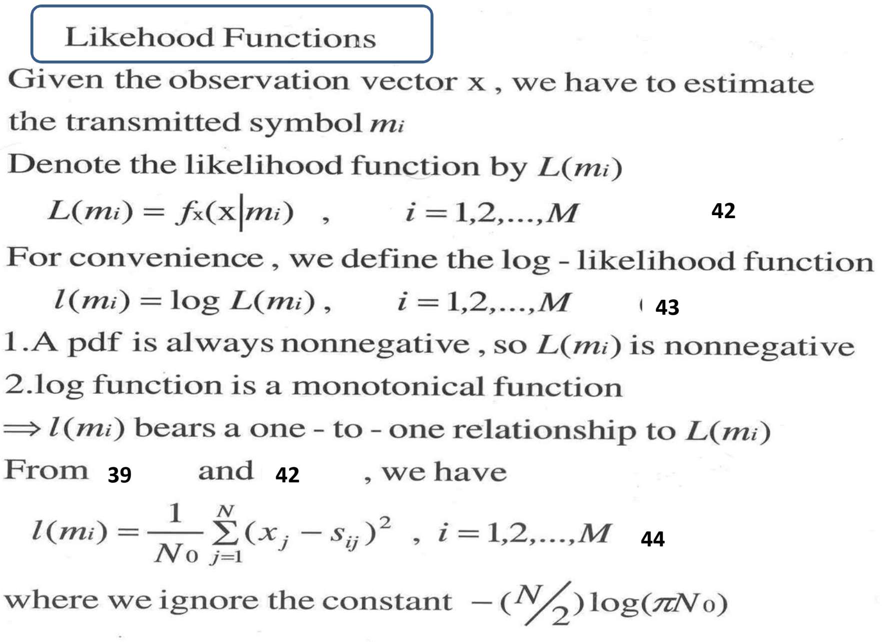
$\Rightarrow$ Theorem of irrelevance: Only the projections of the noise onto the basis functions affects the detections, the remainder is irrelevant

$\Rightarrow$ The AWGN channel is equivalent to an N - dim. vector channel

$$\mathrm{x} = s_i + \mathrm{w} \quad , \quad i = 1,2,\ldots,M$$ 41

Theorem of Irrelevance:

Insofar as signal detection in additive white Gaussian noise is concerned, only the projections of the noise onto the basis functions of the signal set affects the sufficient statistics of the detection problem; the remainder of the noise is irrelevant.



Likehood Functions

Given the observation vector x , we have to estimate the transmitted symbol m_i

Denote the likelihood function by $L(m_i)$

$$L(m_i) = f_x(\mathrm{x}|m_i) \quad , \qquad i = 1,2,\ldots,M \qquad 42$$

For convenience , we define the log - likelihood function

$$l(m_i) = \log L(m_i) \, , \qquad i = 1,2,\ldots,M \qquad 43$$

1.A pdf is always nonnegative , so $L(m_i)$ is nonnegative

2.log function is a monotonical function

$\Rightarrow l(m_i)$ bears a one - to - one relationship to $L(m_i)$

From 39 and 42 , we have

$$l(m_i) = \frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2 \;\; , \;\; i = 1,2,\ldots,M \qquad 44$$

where we ignore the constant $-(N/2)\log(\pi N_0)$

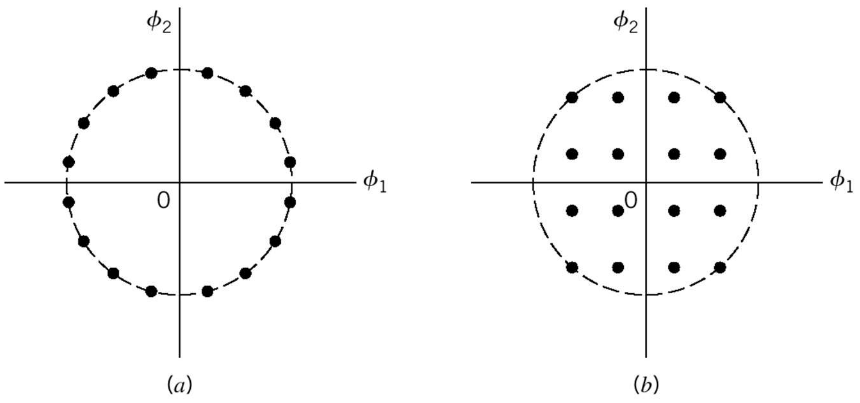
Transmitted signal representations

- Let one of the M possible signals of $s_i(t)$, i=1,2,…M be transmitted in each time slot T with equal probability 1/M.
- This is applied to a bank of correlators supplied with N orthogonal basis functions.
- The resulting correlator o/ps define the signal vector $\mathbf{s_i}$.
- $\mathbf{s_i}$ is represented as a point in Euclidean space of dimensions N<=M, called transmitted signal point.
- The set of message points corresponding to the set of transmitted signals is called Signal Constellation.

Coherent Detection of Signals in Noise:

Maximum Likelihood Decoding.

The set of transmitted signals is called a signal constellation

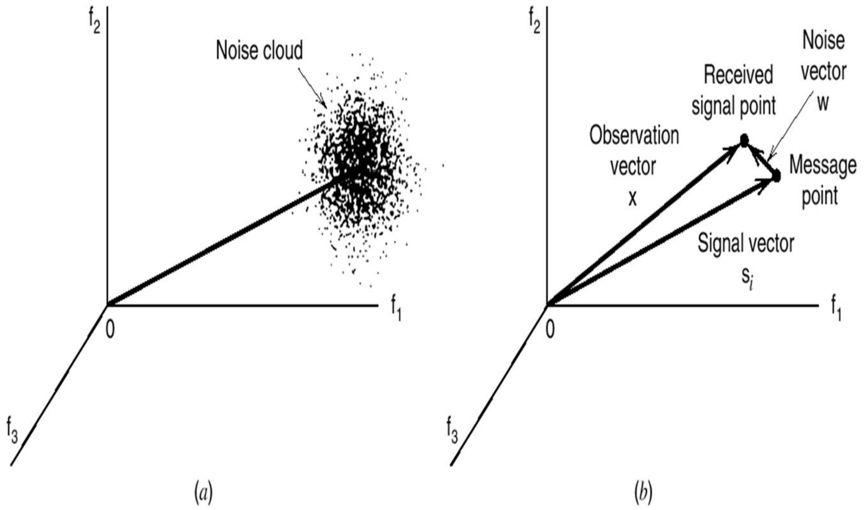


(a) (b)

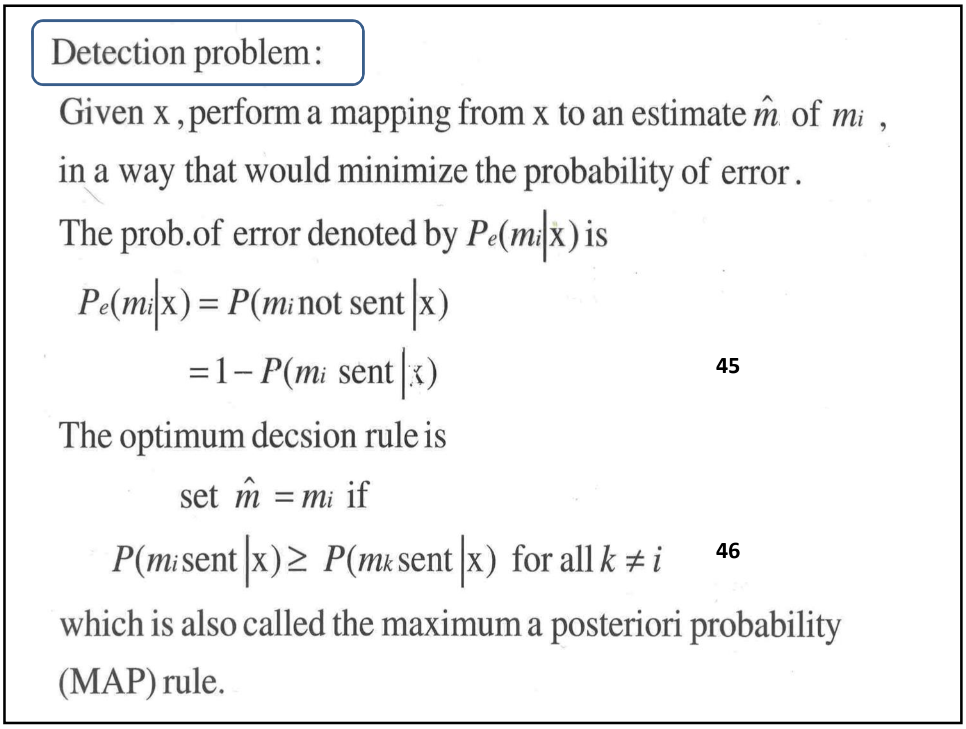
ignal constellation for (a) M-ary PSK and (b) corresponding M-ary QAM, for M = 16.

Representation of received signal x(t)

- Complicated due to presence of w(t).
- Correlator o/ps define observation vector **x** that may be represented in the same Euclidean space as received signal point.
- The received signal point wanders about the message point in a random way and may lie anywhere inside a Gaussian distributed "cloud" with message point as centroid.
- This is due to the **Noise Perturbation** effect.



Illustrating the effect of noise perturbation, depicted in (*a*), on the location of the received signal point, depicted in (*b*).



Detection problem:

Given x, perform a mapping from x to an estimate $\hat{m}$ of m_i, in a way that would minimize the probability of error.

The prob.of error denoted by $P_e(m_i|\mathrm{x})$ is

$$P_e(m_i|\mathrm{x}) = P(m_i \text{ not sent}|\mathrm{x})$$

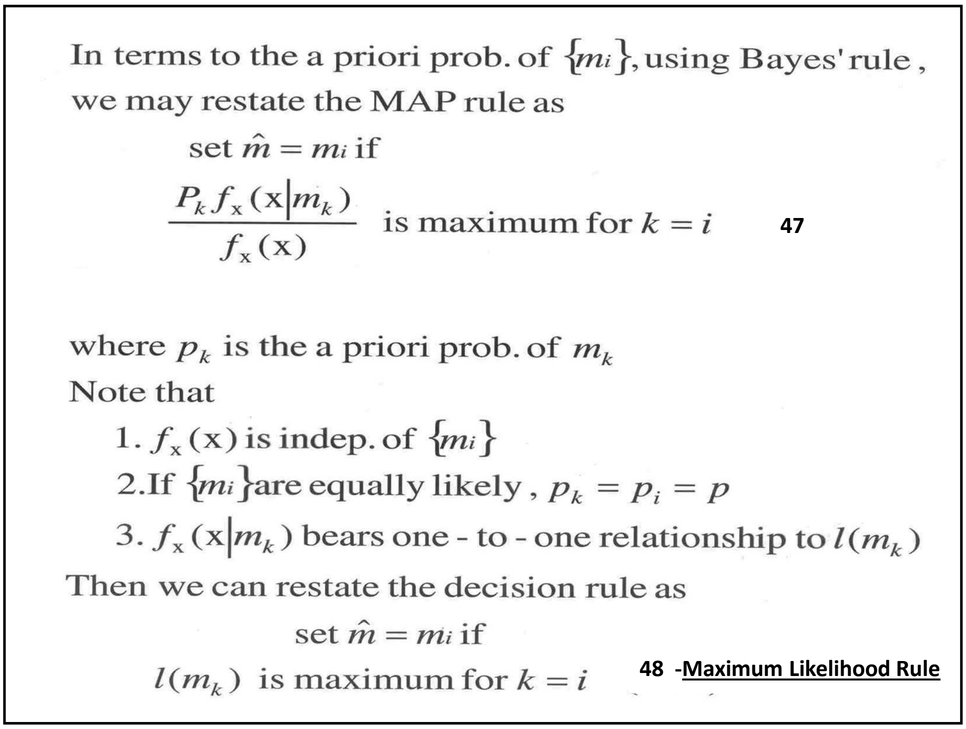
$$= 1 - P(m_i \text{ sent}|\mathrm{x}) \quad 45$$

The optimum decsion rule is

set $\hat{m} = m_i$ if

$$P(m_i \text{ sent}|\mathrm{x}) \geq P(m_k \text{ sent}|\mathrm{x}) \text{ for all } k \neq i \quad 46$$

which is also called the maximum a posteriori probability (MAP) rule.



In terms to the a priori prob. of $\{m_i\}$, using Bayes' rule, we may restate the MAP rule as

set $\hat{m} = m_i$ if

$$\frac{P_k f_{\mathrm{x}}(\mathrm{x}|m_k)}{f_{\mathrm{x}}(\mathrm{x})} \text{ is maximum for } k = i \quad 47$$

where p_k is the a priori prob. of m_k

Note that

1. $f_{\mathrm{x}}(\mathrm{x})$ is indep. of $\{m_i\}$
2. If $\{m_i\}$ are equally likely, $p_k = p_i = p$
3. $f_{\mathrm{x}}(\mathrm{x}|m_k)$ bears one - to - one relationship to $l(m_k)$

Then we can restate the decision rule as

set $\hat{m} = m_i$ if

$l(m_k)$ is maximum for $k = i$ **48 -<u>Maximum Likelihood Rule</u>**

The maximum likelihood decoder differs from the maximum a posteriori decoder (Assum. of p_k = constant)

Let Z denote the N - dim space (observation space).

We may partition Z into M - decision regions denoted by $Z_1, Z_2, ..., Z_M$

Observation vector x lies in Z_i if

$$l(m_k) \text{ is max. for } k = i \qquad 49$$

Recall $$l(m_k) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - \underset{\uparrow}{s_{kj}})^2 \text{ , i} = 1,2,...,M$$

Minimize this term to maximize $l(m_k)$ by the choice $i = k$

x lies in Z_i if

$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \|x - s_k\|^2 \text{ is min. for } k = i \qquad 50$$

$$\Rightarrow x \in Z_i \text{ , if } \|x - s_k\| \text{ is min. for } k = i \qquad 51$$

$\Rightarrow$ to choose the message point closest to the received signal point.

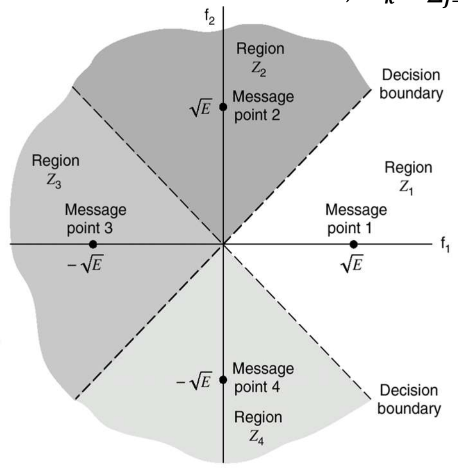
$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \underbrace{\sum_{j=1}^{N} x_j^2}_{\text{indep. of } k} - 2\sum_{j=1}^{N} x_j s_{kj} + \underbrace{\sum_{j=1}^{N} s_{kj}^2}_{\text{energy of } s_k(t) = E_k} \qquad 52$$

Equivalently we have

$$x \in Z_i \text{ if}$$

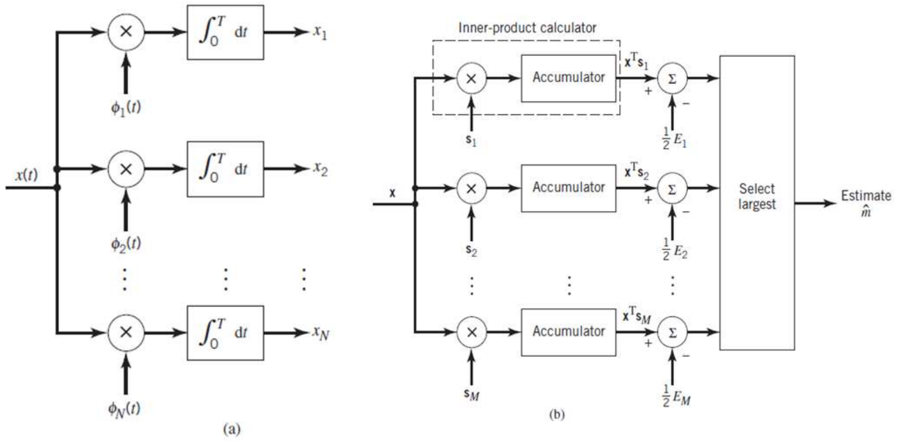
$$\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2} E_k \text{ is max. for } k = i \qquad 53$$

$$where, \quad \boldsymbol{E_k} = \sum_{j=1}^{N} \boldsymbol{s_{kj}}^2 \qquad 54$$



Illustrating the partitioning of the observation space into decision regions for the case when N =2 and M =4; it is assumed that the M transmitted symbols are equally likely.

Overall Correlation Receiver structure



Decoder

Demodulator **Maximum Likelihood ~~Detector~~ (MLD)**

correction!

Correlation Receiver

- The optimum receiver for an AWGN channel and for the case when the transmitted signals $s_1(t)$, $s_2(t)$, …, $s_M(t)$ are equally likely is called a *correlation receiver*.
- It consists of two subsystems, as shown.
- **1.** *Detector*, which consists of *M correlators* supplied with a set of orthonormal basis functions $\phi_1(t)$, $\phi_1(t)$, …, $\phi_N(t)$, that are generated locally; this bank of correlators operates on the received signal $x(t)$, $0 \leq t \leq T$, to produce the observation vector **x**.
- **2.** *Maximum-likelihood decoder*, which operates on the observation vector **x** to produce an estimate of the transmitted symbol *mi, i* = 1, 2, …, *M*, in such a way that the average probability of symbol error is minimized.

- In accordance with the maximum likelihood decision rule, the decoder multiplies the *N* elements of the observation vector **x** by the corresponding *N* elements of each of the *M* signal vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$.
- Then, the resulting products are successively summed in *accumulators* to form the corresponding set of inner products $\{\mathbf{x}^T\mathbf{s}_k | k = 1, 2, \ldots, M\}$.
- Next, the inner products are corrected for the fact that the transmitted signal energies may be unequal.
- Finally, the largest one in the resulting set of numbers is selected, and an appropriate decision on the transmitted message is thereby made.

Matched Filter Receiver

- The optimum detector involves a set of correlators.
- Alternatively, use a different but equivalent structure in place of the correlators.
- To explore this alternative method of implementing the optimum receiver, consider a linear time-invariant filter with impulse response $h_j(t)$.
- With the received signal $x(t)$ operating as input, the resulting filter output is defined by the convolution integral,

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)h_j(t-\tau)\,d\tau \qquad \ldots\ldots\ (55)$$

- To proceed further, evaluate this integral over the duration of a transmitted symbol, namely $0 \leq t \leq T$.
- With time t restricted in this manner, replace the variable τ with t,

$$y_j(T) = \int_0^T x(t)h_j(T-t)\,\mathrm{d}t \quad \text{....... (56)}$$

- Consider next a detector based on a bank of correlators.
- The output of the jth correlator is defined by :

$$x_j = \int_0^T x(t)\phi_j(t)\,\mathrm{d}t \quad \text{....... (57)}$$

- For $y_j(T)$ to equal x_j, from eqns(56) and (57) that this condition is satisfied provided that,

$$h_j(T-t) = \phi_j(t) \quad \text{for } 0 \leq t \leq T \text{ and } j = 1, 2, \ldots, M$$

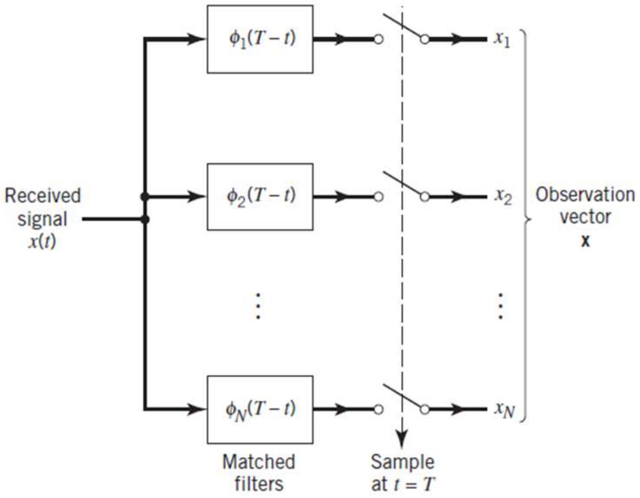
- Equivalently, express the condition imposed on the desired impulse response of the filter,

$$h_j(t) = \phi_j(T-t), \quad \text{for } 0 \leq t \leq T \text{ and } j = 1, 2, \ldots, M \quad \text{....... (58)}$$

- Given a pulse signal $\phi(t)$ occupying the interval $0 \leq t \leq T$, a linear time-invariant filter is said to be matched to the signal $\phi(t)$ if its impulse response $h(t)$ satisfies the condition,

$$h(t) = \phi(T-t) \text{ for } 0 \leq t \leq T \quad \text{....... (59)}$$

A time-invariant filter defined in this way is called a Matched filter. Correspondingly, an optimum receiver using Matched filters in place of Correlators is called a Matched-filter receiver.



Detector part of matched filter receiver

Digital Modulation Schemes

EC302 DC Module IV

Introduction

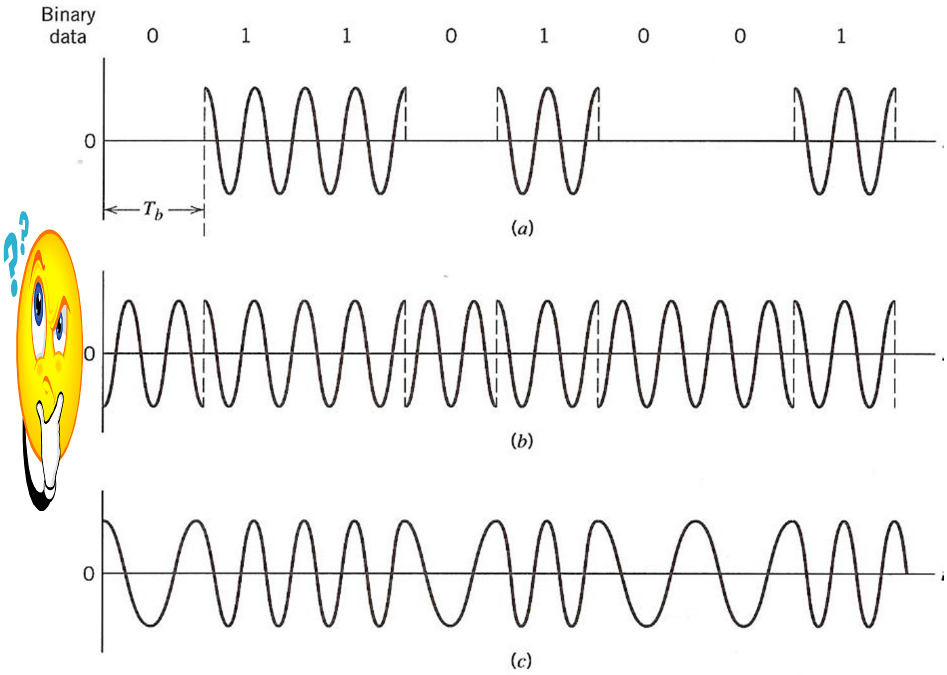
- In baseband pulse transmission, a data stream represented in the form of a discrete pulse-amplitude modulated (PAM) signal is transmitted directly over a low-pass channel.
- In digital passband transmission, on the other hand, the incoming data stream is modulated onto a carrier (usually sinusoidal) with fixed frequency limits imposed by a band-pass channel of interest.
- The communication channel used for passband data transmission may be a microwave radio link, a satellite channel, or the like.
- The modulation process making the transmission possible involves switching (keying) the amplitude, frequency, or phase of a sinusoidal carrier in some fashion in accordance with the incoming data.
- Thus there are three basic signaling schemes, and they are amplitude-shift keying (ASK), frequency-shift keying (FSK), and phase-shift keying (PSK).
- They may be viewed as special cases of amplitude modulation, frequency modulation, and phase modulation.

- In the transmission of digital info over communication channel, Modulator is the interface device that maps digital information into analog waveforms that match channel characteristics.
- Mapping takes blocks of $k=\log_2 M$ bits at a time from the info sequence $\{a_n\}$ and selects one of $M=2^k$ deterministic, finite energy waveforms $\{s_m(t), m=1,2,\ldots,M\}$ for tx over the channel.

- Mapping done under constraint that a waveform transmitted in any time interval depends on one or more previously transmitted waveforms → Modulator has Memory.

- Otherwise, the Modulator is said to be Memoryless.

- Data transmission uses a sine carrier wave modulated by data stream.
- Modulation involves switching amplitude, frequency, or phase of a carrier in accordance with the incoming data.
- There are three basic signaling schemes:
  - Amplitude-shift keying (ASK)
  - Frequency-shift Keying (FSK)
  - Phase-shift keying (PSK)

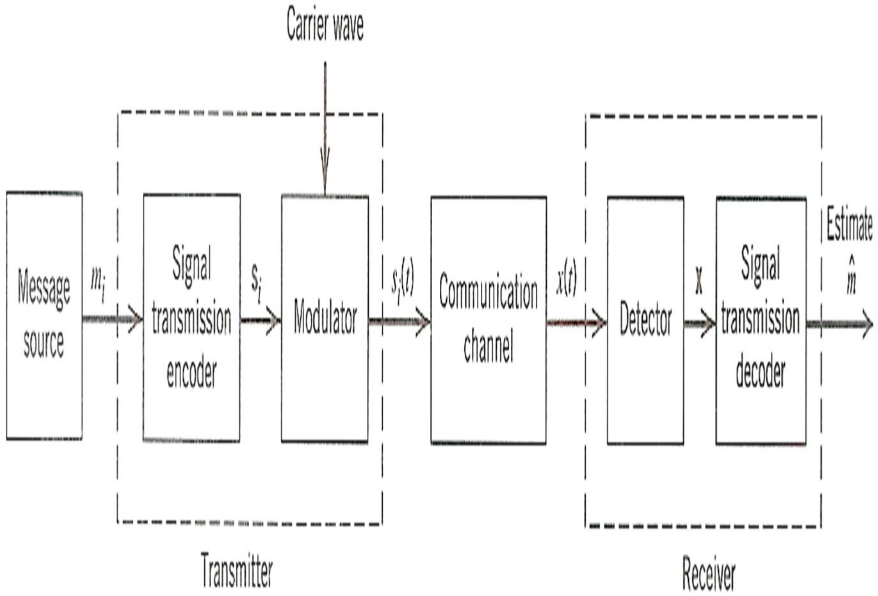


- Unlike ASK signals, both PSK and FSK signals have a const envelope.
- This property makes PSK and FSK signals *impervious to amplitude nonlinearities*.
- In practice, PSK and FSK signals are preferred to ASK signals for *passband data Tx over nonlinear channels*.
- Digital modulation techniques may be classified into *coherent* and *noncoherent* techniques, depending on whether Rx is equipped with a *phase-recovery circuit* or not.
- *Phase-recovery circuit* ensures osc supplying locally gen carrier in Rx is ***synchronized*** (in both frequency and phase) to osc supplying carrier used to originally modulate incoming data stream in Tx.

Passband Transmission Model

- In a functional sense, model a passband data transmission system as shown→
- First, a message source exists that emits one symbol every T seconds, with the symbols belonging to an alphabet of M symbols, which we denote by m_1, m_2 …m_M.
- The a priori probabilities $P(m_1)$, $P(m_2)$, …, $P(m_M)$ specify the message source output.
- When the M symbols of the alphabet are equally likely,

$$p_i = P(m_i) \quad = \frac{1}{M} \quad \text{for all } i \qquad \text{……….(1)}$$



Functional model of passband data transmission system

- The M-ary output of the message source is presented to a <u>signal transmission encoder</u>, producing a corresponding vector $\mathbf{s}_i$ made up of N real elements, one such set for each of the M symbols of the source alphabet.
- Note that the dimension $N \leq M$.
- With the vector $\mathbf{s}_i$ as input, the <u>modulator</u> builds a distinct signal $s_i(t)$ of duration T seconds as the representation of the symbol m_i generated by the message source.
- The signal $s_i(t)$ is necessarily an energy signal,

$$E_i = \int_0^T s_i^2(t)\, dt, \qquad i = 1, 2, \ldots, M \qquad \text{……….(2)}$$

- The bandpass communication channel, coupling the transmitter to the receiver, is assumed to have two characteristics:
  - **1**. The channel is linear, with a bandwidth that is wide enough to accommodate the transmission of the modulated signal $s_i(t)$ with negligible or no distortion.
  - **2**. The channel noise w(t) is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$.
- The receiver, which consists of a detector followed by a signal transmission decoder, performs two functions:
  - ❖ **1**. It reverses the operations performed in the transmitter.
  - ❖ **2**. It minimizes the effect of channel noise on the estimate $\hat{m}$ computed for the trans mitted symbol m_i.

Binary Phase-Shift Keying

- In a coherent binary PSK system, signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0.
- A pair of sinusoidal waves that differ only in a relative phase shift of 180^o are called antipodal signals.

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \qquad \text{..........(3)}$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \qquad \text{..........(4)}$$

where $0 \le t \le T_b$ and E_b is the transmitted signal energy per bit.

- To ensure that each transmitted bit contains an integral no: of cycles of carrier wave, f_c is made equal to n_c/T_b for some fixed integer n_c.
- In the case of binary PSK, there is <u>only one basis function</u> of unit energy:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), \quad 0 \le t < T_b \qquad \text{……….(5)}$$

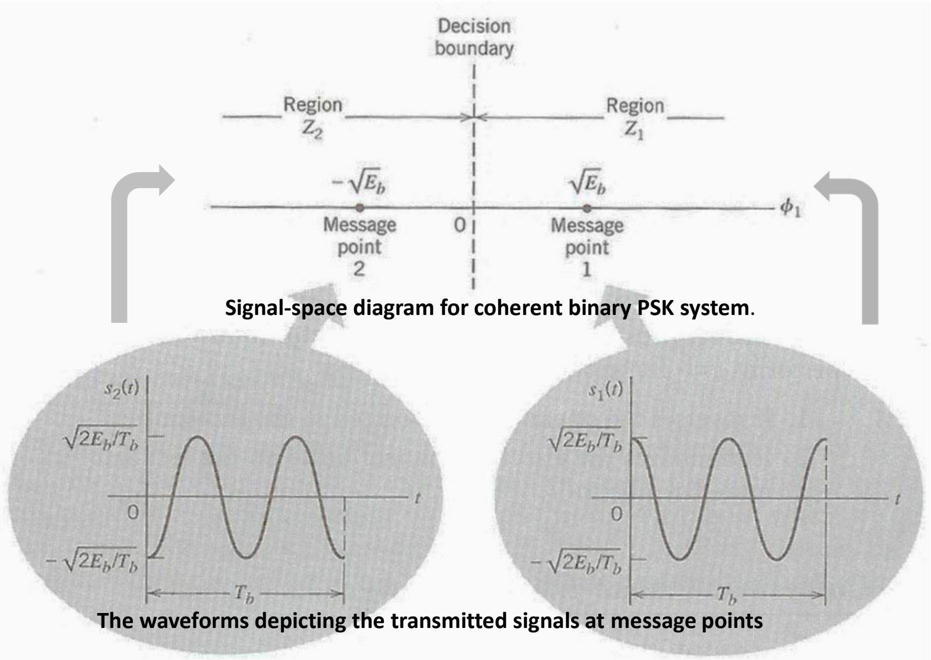
……….(6)

$$s_1(t) = \sqrt{E_b}\,\phi_1(t),\ s_2(t) = -\sqrt{E_b}\,\phi_1(t),\ 0 \le t < T_b$$

$$s_{11} = \int_0^{T_b} s_1(t)\phi_1(t)\,dt = +\sqrt{E_b} \qquad s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\,dt = -\sqrt{E_b}$$

……….(7)

Phase-modulated signals (Binary Phase-Shift Keying)



Signal-space diagram for coherent binary PSK system.

The waveforms depicting the transmitted signals at message points

Coherent detection

- Partition the signal space into two regions:
- set of points closest to message point 1 at $+\sqrt{E_b}$;
- set of points closest to message point 2 at $-\sqrt{E_b}$;
- construct the midpoint of the line joining these two message points and then marking off the appropriate decision regions.
- these two decision regions are marked Z_1 and Z_2, according to the message point around which they are constructed.

- The decision rule is now simply to decide that signal $s_1(t)$ (i.e., binary symbol 1) was transmitted if the received signal point falls in region Z_1 …
- …and to decide that signal $s_2(t)$ (i.e., binary symbol 0) was transmitted if the received signal point falls in region Z_2.
- Two kinds of erroneous decisions may, however, be made:
  - **1**. *Error of the first kind*. Signal $s_2(t)$ is transmitted but the noise is such that the received signal point falls inside region Z_1; so the receiver decides in favor of signal $s_1(t)$.
  - **2.** *Error of the second kind*. Signal $s_1(t)$ is transmitted but the noise is such that the received signal point falls inside region Z_2; so the receiver decides in favor of signal $s_2(t)$.

BPSK-Probability of Bit Error

- To calculate the probability of making an error of the first kind, note that the decision region associated with symbol 1 or signal $s_1(t)$ is described by,

$$Z_1 : 0 < x_1 < \infty$$

- where the observable element x_1 is related to the received signal $x(t)$ by,

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\, dt \qquad \text{..........(8)}$$

- The conditional probability density function of random variable X_1, given that symbol 0 (i.e., signal $s_2(t)$was transmitted, is defined by,

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right] \qquad \text{..........(9)}$$

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] \qquad \text{..........(10)}$$

- The conditional probability of the receiver deciding in favor of symbol 1, given that symbol 0 was transmitted,

$$p_{10} = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1 \qquad \text{..........(11)}$$

- Putting

$$z = \sqrt{\frac{2}{N_0}}(x_1 + \sqrt{E_b})$$

- and changing the variable of integration from *x*1 to *z*, we may compactly rewrite eqn(11) in terms of the *Q*-function:

$$p_{10} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E_b/N_0}}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz \qquad \text{……….(12)}$$

- The related function commonly used in the context of communication is *Q-function*, which is formally defined as,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \qquad \text{……….(13)}$$

$$p_{10} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \qquad \text{……….(14)}$$

- Consider next an error of the second kind.
- Note that the signal space of is symmetric with respect to the origin.
- It follows, therefore, that p_{01}, the conditional probability of the receiver deciding in favor of symbol 0, given that symbol 1 was transmitted, also has the same value as in eqn(14).

- Thus, averaging the conditional error probabilities p_{10} and p_{01}, we find that the *average* Probability of symbol error or, equivalently, the BER for binary PSK using coherent detection and assuming equiprobable symbols is given by,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \qquad \text{……….(15)}$$

Phase-modulated signals (Binary Phase-Shift Keying)

- **Probability of Bit Error** is proportional to the distance between the closest points in the constellation.
  - A simple upper bound can be found using the assumption that noise is additive, white, and Gaussian.

$$\text{Prob}\{\text{bit error}\} \le Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

  - *d* is distance between nearest constellation points.

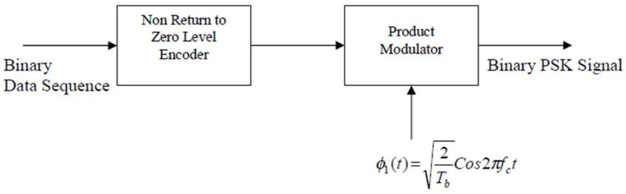
Probability of Bit Error - BPSK

- *Q(x)* is the **Q-function**, the area under a normalized Gaussian function (also called a Normal curve or a bell curve)

$$Q(z) = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$d = 2\sqrt{E_b} \quad \text{so} \quad \text{Prob}\{\text{bit error}\} \le Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Coherent Binary PSK:



Fig(a) Block diagram of BPSK transmitter

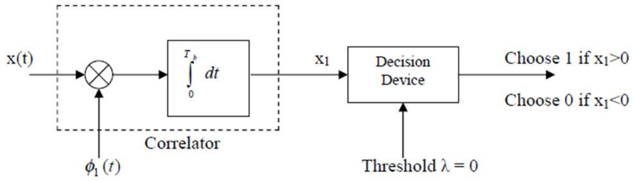


Fig (b) Coherent binary PSK receiver

BPSK TRANSMITTER

- The transmitter consists of two components:
- **1.** Polar NRZ-level encoder, which represents symbols 1 and 0 of the incoming binary sequence by amplitude levels.
- **2.** Product modulator, which multiplies the output of the polar NRZ encoder by the basis function $\phi_1(t)$; in effect, the sinusoidal $\phi_1(t)$ acts as the "carrier" of the binary PSK signal.

BPSK RECEIVER

- To make an optimum decision on the received signal $x(t)$ in favor of symbol 1 or symbol 0 (i.e., estimate the original binary sequence at the transmitter input), assume that the receiver has access to a locally generated replica of the basis function $\phi_1(t)$.
- In other words, the receiver is synchronized with the transmitter, as shown in the block diagram of Fig b.
- Identify two basic components in the binary PSK receiver:
  - **1.** Correlator, which correlates the received signal $x(t)$ with the basis function $\phi_1(t)$ on a bit-by-bit basis.
  - **2.** Decision device, which compares the correlator output against a zero-threshold, assuming that binary symbols 1 and 0 are equi-probable.
  - If the threshold is exceeded, a decision is made in favor of symbol 1; if not, the decision is made in favor of symbol 0.
- Equality of the correlator with the zero-threshold is decided by the toss of a fair coin (i.e., in a random manner).

Binary Frequency-Shift Keying (BFSK)

- In *binary FSK*, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency by a fixed amount.
- A typical pair of sinusoidal waves is given by,

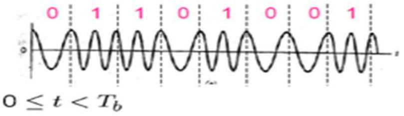
$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_i t), & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \qquad \text{..........(16)}$$

where $i = 1, 2$ and E_b is the transmitted signal energy per bit;

- the transmitted frequency is set at,

$$f_i = \frac{n_c + 1}{T_b} \qquad \text{..........(17)}$$

- for some fixed integer n_c and $i = 1, 2$
- Symbol 1 is represented by $s_1(t)$ and symbol 0 by $s_2(t)$.
- This FSK signal is known as Sunde's FSK.
- It is a continuous-phase signal, in the sense that phase continuity is always maintained, including the inter-bit switching times.



- From eqns(16) and (17), observe directly that the signals $s_1(t)$ and $s_2(t)$ are orthogonal, but not normalized to have unit energy.
- The most useful form for the set of orthonormal basis functions is,

$$\phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_i t), & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \quad \text{..........(18)}$$

where *i* = 1, 2.

- coefficient s_{ij} for where *i* = 1, 2 and *j* = 1, 2 is,

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\,dt$$

$$= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_i t)\sqrt{\frac{2}{T_b}}\cos(2\pi f_j t)\,dt \quad \text{..........(19)}$$

$$s_{ij} = \begin{cases} \sqrt{E_b}, & i = j \\ 0, & i \ne j \end{cases} \quad \text{..........(20)}$$

- Thus, unlike binary PSK, binary FSK is characterized by having a signal-space diagram that is two-dimensional (i.e., *N* = 2) with two message points (i.e., *M* = 2).
- The two message points are defined by the vectors,

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad \text{..........(21)} \qquad \mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \quad \text{..........(22)}$$

- The Euclidean distance,

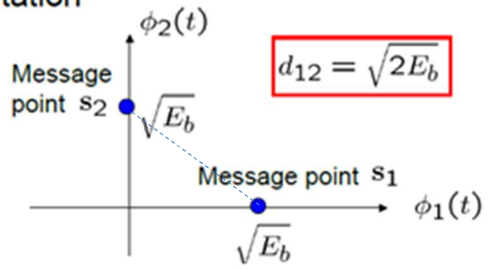
$$\|\mathbf{s}_1 - \mathbf{s}_2\| = \sqrt{2E_b}. \quad \text{..........(23)}$$

BFSK SIGNAL SPACE REPRESENTATION

Signal space representation

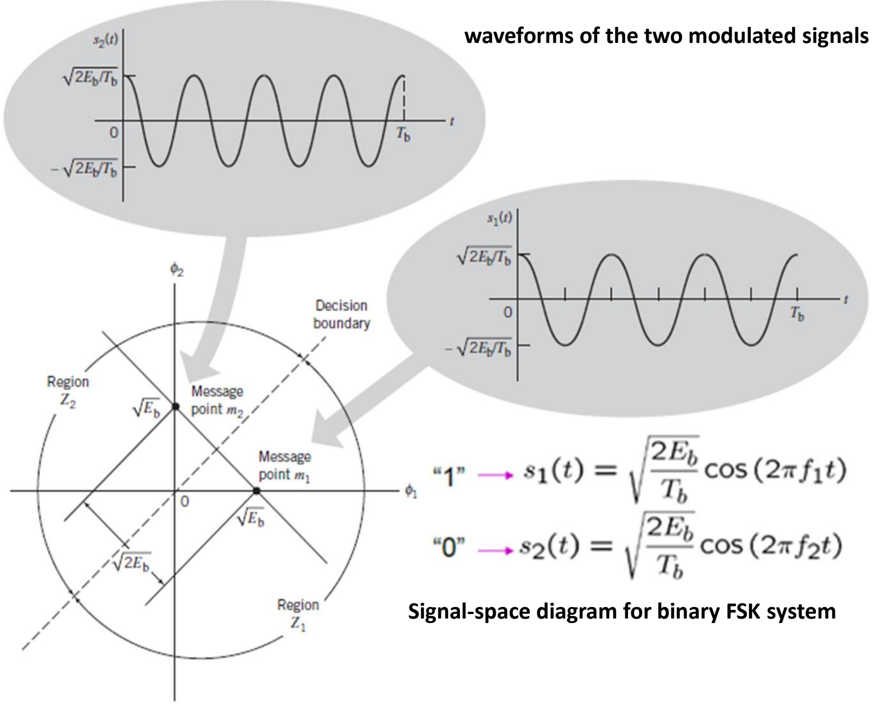
$$s_1 = [\sqrt{E_b} \quad 0]$$

$$s_2 = [0 \quad \sqrt{E_b}]$$



$$\boxed{d_{12} = \sqrt{2E_b}}$$

Apply Pythagoras Theorem,
$d_{12} = \sqrt{(E_b + E_b)} = \sqrt{2E_b}$



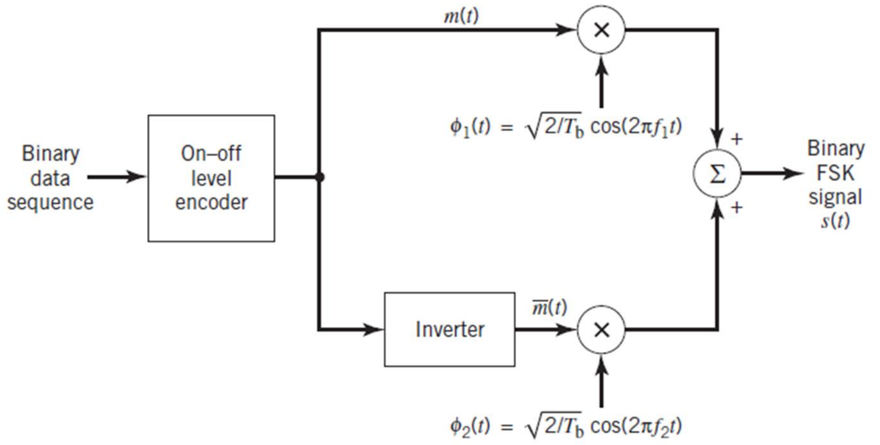
waveforms of the two modulated signals

$$\text{“1”} \longrightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)$$

$$\text{“0”} \longrightarrow s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$$

Signal-space diagram for binary FSK system

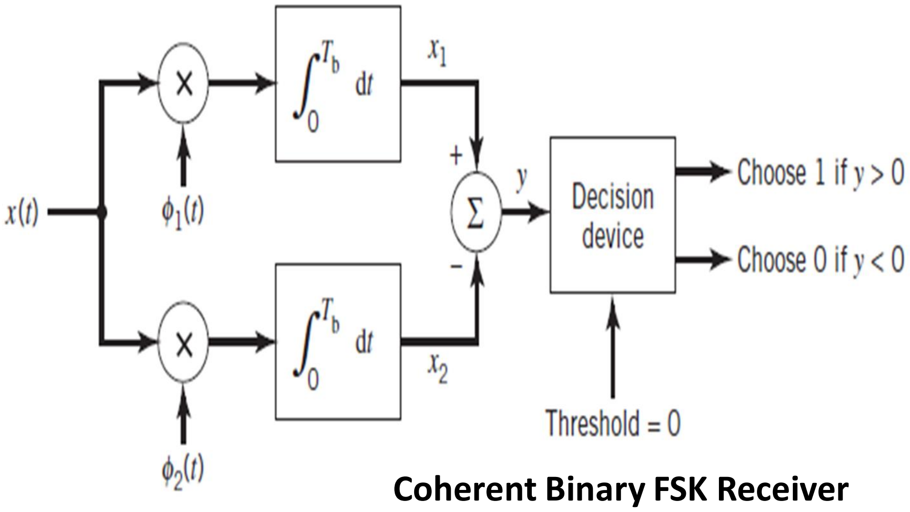
Generation of Binary FSK Signals



Coherent Binary FSK Transmitter

- The block diagram for generating the binary FSK signal consists of two components:
- **1.** On–off level encoder, the output of which is a constant amplitude of $\sqrt{E_b}$ in response to input symbol 1 and zero in response to input symbol 0.
- **2.** Pair of oscillators, whose frequencies f_1 and f_2 differ by an integer multiple of the bit rate $1/T_b$ in accordance with eqn (17).
  - The lower oscillator with frequency f_2 is preceded by an inverter.
  - When in a signaling interval, the input symbol is 1, the upper oscillator with frequency f_1 is switched on and signal $s_1(t)$ is transmitted, while the lower oscillator is switched off.
  - On the other hand, when the input symbol is 0, the upper oscillator is switched off, while the lower oscillator is switched on and signal $s_2(t)$ with frequency f_2 is transmitted.
  - With phase continuity as a requirement, the two oscillators are *synchronized* with each other.
  - Alternatively, use a voltage-controlled oscillator (VCO), in which case phase continuity is automatically satisfied.

Coherent Detection of Binary FSK Signals



Coherent Binary FSK Receiver

- To coherently detect the original binary sequence given the noisy received signal $x(t)$, use the receiver shown.
- It consists of two correlators with a common input, which are supplied with locally generated coherent reference signals $\phi_1(t)$ and $\phi_2(t)$.
- The correlator outputs are then subtracted, one from the other and the resulting difference y is then compared with a threshold of zero.
- If $y > 0$, the receiver decides in favor of 1.
- On the other hand, if $y < 0$, it decides in favor of 0.
- If y is exactly zero, the receiver makes a random guess (i.e., flip of a fair coin) in favor of 1 or 0.

Error Probability of Binary FSK

- The observation vector **x** has two elements x_1 and x_2 that are defined by,

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\,dt \quad \text{..........(24)} \qquad x_2 = \int_0^{T_b} x(t)\phi_2(t)\,dt \quad \text{..........(25)}$$

- where $x(t)$ is the received signal, whose form depends on which symbol was transmitted.
- Given that symbol 1 was transmitted, $x(t)$ equals $s_1(t) + w(t)$, where $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$.
- If, on the other hand, symbol 0 was transmitted, $x(t)$ equals $s_2(t) + w(t)$.

- the observation space is partitioned into two decision regions, Z_1 and Z_2.
- The decision boundary, separating region Z_1 from region Z_2, is the perpendicular bisector of the line joining the two message points.
- The receiver decides in favor of symbol 1 if received signal point represented by the observation vector **x** falls inside region Z_1. This occurs when $x_1 > x_2$.
- If, on the other hand, we have $x_1 < x_2$, the received signal point falls inside region Z_2 and the receiver decides in favor of symbol 0.
- On the decision boundary, $x_1 = x_2$, in which case the receiver makes a random guess in favor of symbol 1 or 0.
- define a new Gaussian random variable Y whose sample value y is equal to the difference between x_1 and x_1 → $y = x_1 - x_2$ ....(26)

- The mean value of the random variable *Y* depends on which binary symbol was transmitted.
- Given that symbol 1 was sent, the Gaussian random variables X_1 and X_2, whose sample values are denoted by x_1 and x_2, have mean values equal to and zero, respectively.
- Correspondingly, the conditional mean of the random variable *Y* given that symbol 1 was sent is,

$$\mathbb{E}[Y|1] = \mathbb{E}[X_1|1] - \mathbb{E}[X_2|1] = +\sqrt{E_b} \quad \text{……….(27)}$$

- On the other hand, given that symbol 0 was sent, the random variables X_1 and X_2 have mean values equal to zero and , respectively.
- Correspondingly, the conditional mean of the random variable *Y* given that symbol 0 was sent is,

$$\mathbb{E}[Y|0] = \mathbb{E}[X_1|0] - \mathbb{E}[X_2|0] = -\sqrt{E_b} \quad \text{……….(28)}$$

- The variance of the random variable *Y* is <u>independent</u> of which binary symbol was sent.
- Since the random variables X_1 and X_2 are statistically independent, each with a variance of $N_0/2$,

$$\mathrm{var}[Y] = \mathrm{var}[X_1] + \mathrm{var}[X_2] = N_0 \quad \text{……….(29)}$$

- Suppose that symbol 0 was sent.
- The conditional probability density function of the random variable *Y* is then given by,

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(y+\sqrt{E_b})^2}{2N_0}\right] \quad ……….(30)$$

- $x_1 > x_2$ or $y > 0$ corresponds to the receiver making a decision in favor of symbol 1, the conditional probability of error given that symbol 0 was sent,

$$p_{10} = \mathbb{P}(y > 0|\text{symbol 0 was sent}) = \int_0^\infty f_Y(y|0)\, dy$$

$$= \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty \exp\left[-\frac{(y+\sqrt{E_b})^2}{2N_0}\right] dy \quad ……….(31)$$

- To put the integral in eqn(31) in a standard form involving the *Q*-function,

$$\frac{y+\sqrt{E_b}}{\sqrt{N_0}} = z \quad ……….(32)$$

- Then, changing the variable of integration from *y* to *z*, rewrite eqn(31) as,

$$p_{10} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{E_b/N_0}}^\infty \exp\left(-\frac{z^2}{2}\right) dz$$

$$= Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad ……….(33)$$

- Similarly, p_{01}, the conditional probability of error given that symbol 1 was sent, has the same value as in eqn(33).
- averaging p_{10} and p_{01} and assuming equiprobable symbols, the <u>average probability of bit error</u> or, equivalently, the <u>BER for binary FSK using coherent detection</u> is,

⇨ **3 dB** worse than BPSK

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad \text{……….(34)}$$

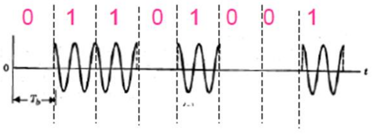
To achieve the same P_e, BFSK needs 3dB more transmission power than BPSK

Binary ASK

- Modulation

“1” $\longrightarrow$ $s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_c t)$

“0” $\longrightarrow$ $s_2(t) = 0 \qquad 0 \le t < T_b$



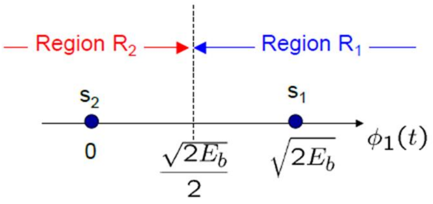
(On-off signaling)

- Average energy per bit

$$E_b = \frac{E + 0}{2} \qquad \text{i.e. } E = 2E_b$$

- Decision Region

$$d_{12} = \sqrt{2E_b}$$



Probability of Error for Binary ASK

- Average probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Identical to that of coherent binary FSK

- Exercise: Prove P_e

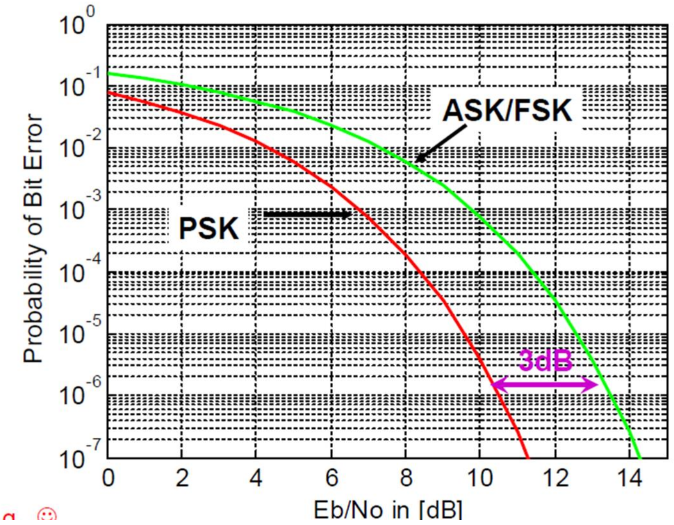
Probability of Error and the Distance Between Signals

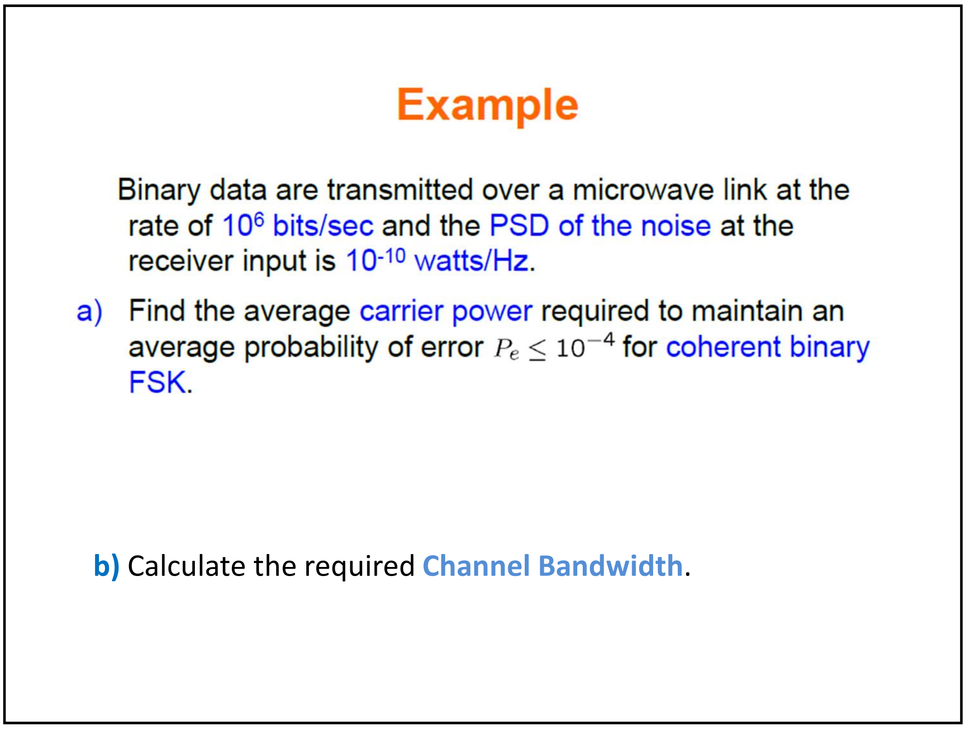
BPSK	BFSK	BASK
$d_{1,2} = 2\sqrt{E_b}$	$d_{1,2} = \sqrt{2E_b}$	$d_{1,2} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

- In general,

$$\boxed{P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)}$$

Probability of Error for BPSK and FSK/ASK



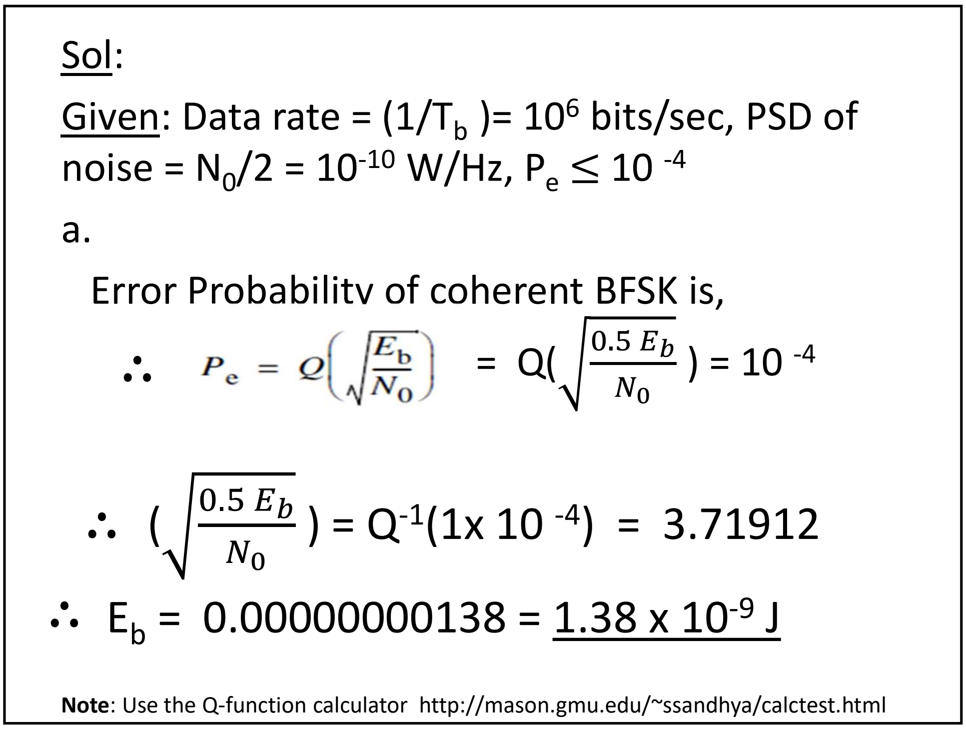


Example

Binary data are transmitted over a microwave link at the rate of 10^6 bits/sec and the PSD of the noise at the receiver input is 10^{-10} watts/Hz.

a) Find the average carrier power required to maintain an average probability of error $P_e \leq 10^{-4}$ for coherent binary FSK.

b) Calculate the required **Channel Bandwidth**.



Sol:

Given: Data rate = $(1/T_b) = 10^6$ bits/sec, PSD of noise = $N_0/2 = 10^{-10}$ W/Hz, $P_e \leq 10^{-4}$

a.

Error Probability of coherent BFSK is,

$$\therefore \quad P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{0.5\,E_b}{N_0}}\right) = 10^{-4}$$

$$\therefore \quad \left(\sqrt{\frac{0.5\,E_b}{N_0}}\right) = Q^{-1}(1\text{x}\,10^{-4}) = 3.71912$$

$$\therefore \quad E_b = 0.00000000138 = \underline{1.38 \text{ x } 10^{-9}\text{ J}}$$

Note: Use the Q-function calculator http://mason.gmu.edu/~ssandhya/calctest.html

∴ Power of bit P = E_b/T_b = 1.38 x 10^{-9} /10^{-6}J/s

= <u>0.00138 W</u>

∵ To achieve $P_e \leq 10^{-4}$, average power of carrier must be $\geq$ <u>0.00138 W</u>

b.

The channel BW is approx. equal to bit rate.

$B_T = 1/T_b = 10^6$ Hz = <u>1 MHz</u>

M-ary signaling scheme

- The no: of possible signals $M=2^n$.
- Symbol duration $T=nT_b$, where T_b is bit duration.
- There are ***M*-ary ASK**, ***M*-ary PSK**, and ***M*-ary FSK**.
- Combine diff methods of modulation into a hybrid form.
- For ex , ***M*-ary amplitude-phase keying(APK)** and ***M*-ary quadrature-amplitude modulation (QAM).**
- ***M*-ary PSK** and ***M*-ary QAM** are examples of *linear modulation.*
- An ***M*-ary PSK** signal has a *const envelope*, whereas an ***M*ary QAM** signal involves *changes in carrier amplitude*.

M-ary signaling scheme

- ***M*-ary PSK** can be used to transmit digital data over a *nonlinear band-pass channel*, whereas ***M*-ary QAM** requires the use of a *linear channel*.
- ***M*-ary PSK**, and ***M*-ary QAM** are commonly used in *coherent systems*.
- **ASK** and **FSK** lend themselves naturally to use in *non-coherent systems* whenever it is impractical to maintain carrier phase synchronization.
- A Non-coherent **PSK** scheme is **DPSK** …

Phase-modulated signals (*M*-ary PSK)

- The *M* signal waveforms are represented as:

$$s_m(t)$$

$$= \mathrm{Re}\left[g(t) e^{j2\pi(m-1)/M} e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T$$

- The signal waveforms have equal energy:

$$\varepsilon = \int_0^T s_m^2(t)\,dt = \frac{1}{2}\int_0^T g^2(t)\,dt = \frac{1}{2}\varepsilon_g$$

- The signal waveforms may be represented as a linear combination of **two orthonormal signal waveforms**:

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t)$$

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}}g(t)\cos 2\pi f_c t \text{ and } f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}}g(t)\sin 2\pi f_c t$$

- Write down the 2D vector: $\mathbf{s_m} = [s_{m1}\ s_{m2}]$?

- Euclidean distance

$$d_{mn} = \|\mathbf{s}_m - \mathbf{s}_n\| = \sqrt{2E_s\left(1 - \cos\frac{2\pi(m-n)}{M}\right)}$$

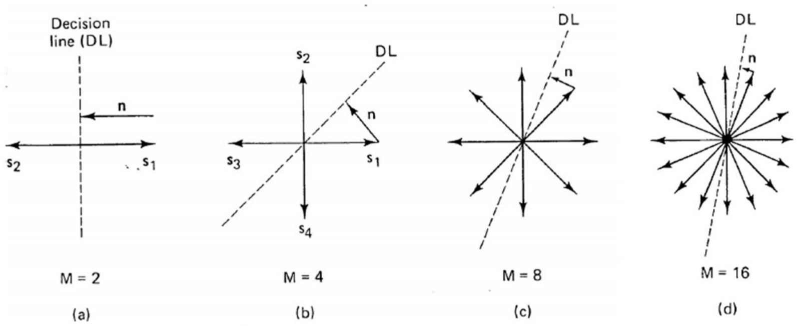
M-ary Modulation

256 QAM

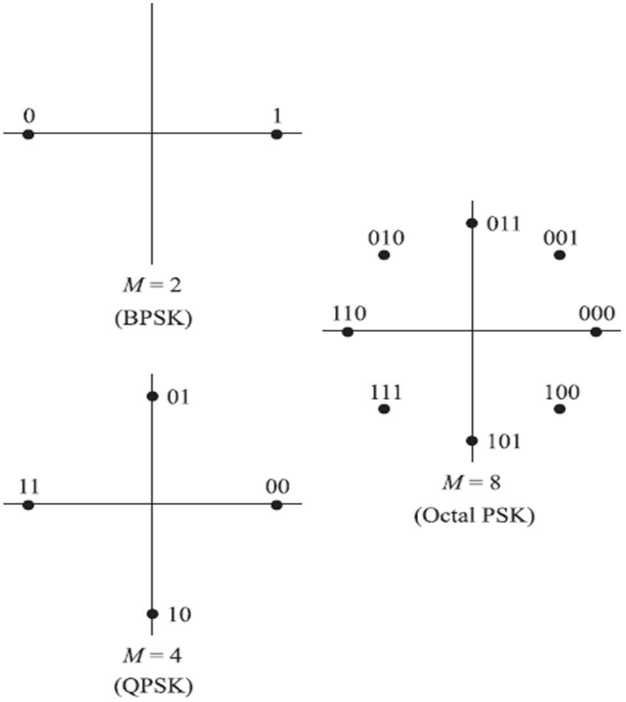
128 QAM

64QAM

MPSK Signal Constellations



BPSK QPSK 8PSK 16PSK



Quadri Phase-Shift Keying

- The need to provide reliable performance, indicated by a very low probability of error, is one important goal in the design of a digital communication system.
- Another important goal is the efficient utilization of channel bandwidth.
- Quadri phase shift keying (QPSK) is a bandwidth-conserving modulation scheme, using coherent detection.

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right], & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2, 3, 4$; E is the transmitted signal energy per symbol, and T is the symbol duration.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[(2i-1)\frac{\pi}{4}\right] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin\left[(2i-1)\frac{\pi}{4}\right] \sin(2\pi f_c t)$$

- Defined a pair of quadrature carriers:

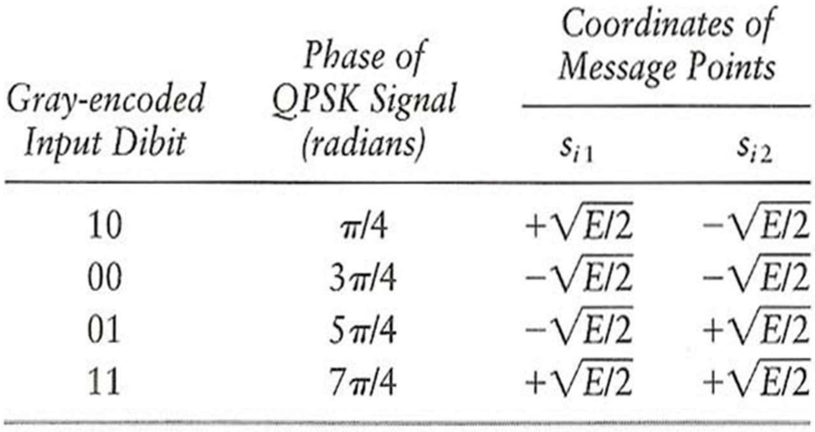
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \qquad 0 \le t \le T$$

- There are **four message points**, and the associated signal vectors are defined by

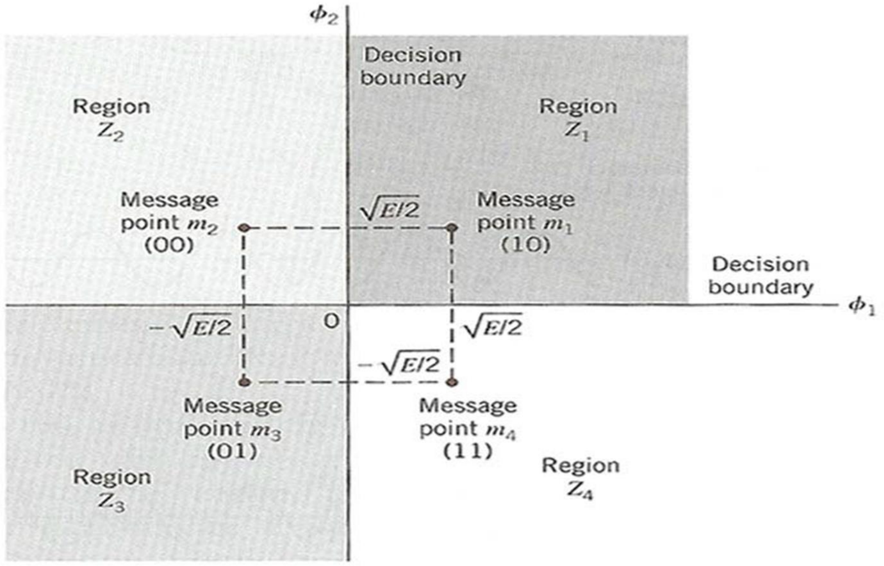
$$s_i(t) = \begin{bmatrix} \sqrt{E} \cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E} \sin\left((2i-1)\frac{\pi}{4}\right) \end{bmatrix}, \qquad i = 1,2,3,4$$

- Each possible value of the phase corresponds to a unique dibit.
- The Gray coding is commonly used.



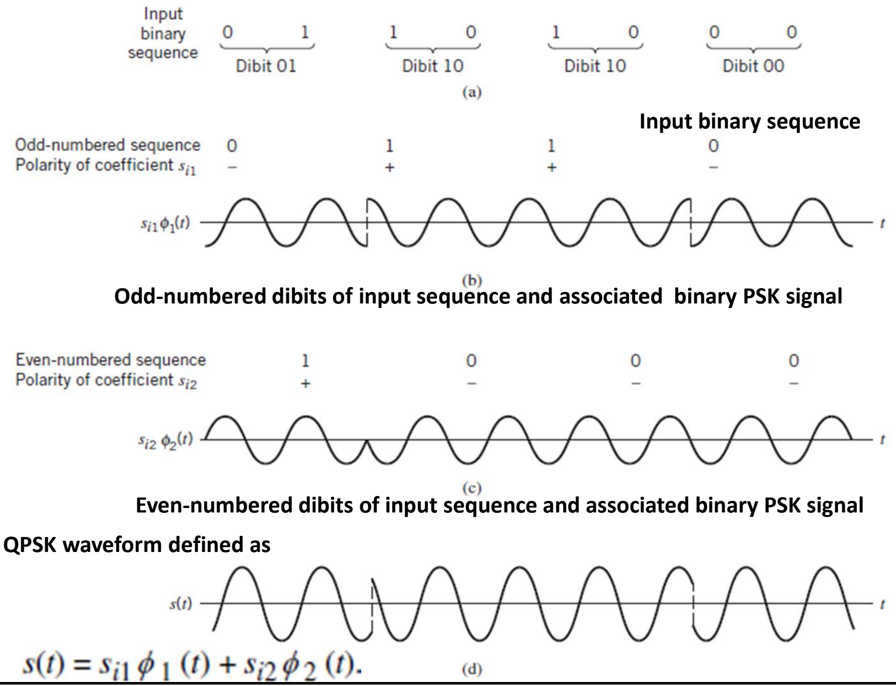
Gray-encoded Input Dibit	Phase of QPSK Signal (radians)	Coordinates of Message Points	
		s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

Signal space diagram of coherent QPSK system



QPSK Waveforms

- Consider the input binary sequence 01101000.
- This sequence is divided into two other sequences, consisting of odd- and even-numbered bits of the input sequence.
- The waveforms representing the two components of the QPSK signal, namely $s_{i1}\phi_1(t)$ and $s_{i2}\phi_2(t)$.
- These two waveforms may individually be viewed as examples of a binary PSK signal.
- Adding them, we get the QPSK waveform.



Input binary sequence

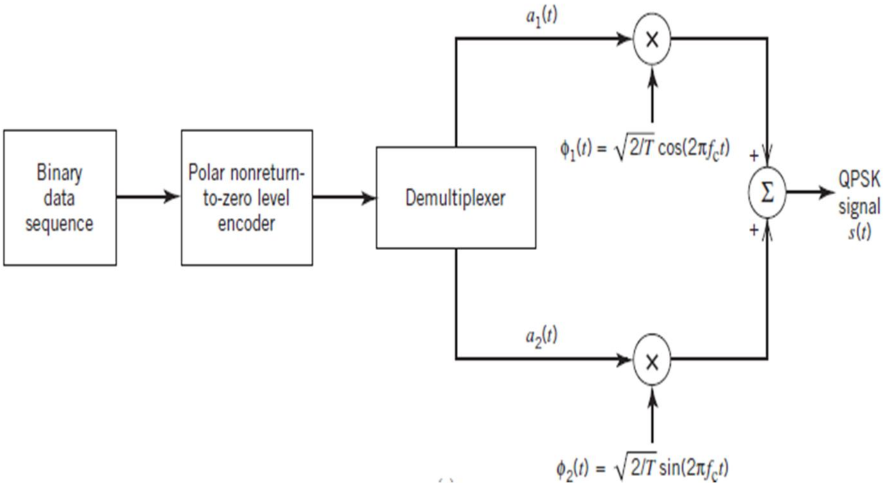
Odd-numbered dibits of input sequence and associated binary PSK signal

Even-numbered dibits of input sequence and associated binary PSK signal

QPSK waveform defined as

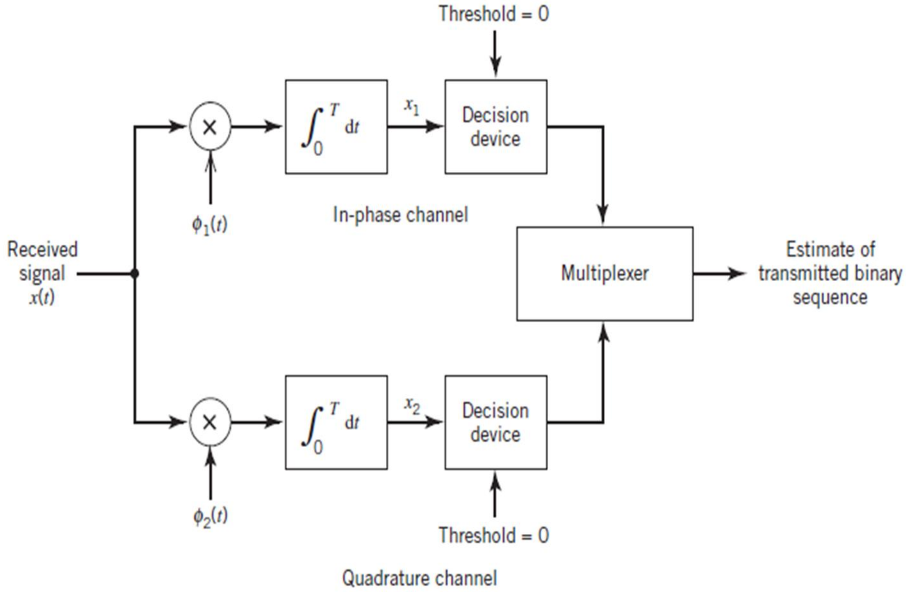
$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t).$$

Generation of QPSK Signals



- A distinguishing feature of the QPSK transmitter is the block labeled <u>Demultiplexer</u>.
- The function of the Demultiplexer is to divide the binary wave produced by the polar NRZ-level encoder into two separate binary waves, one of which represents the <u>odd-numbered dibits</u> in the incoming binary sequence and the other represents the <u>even-numbered dibits</u>.
- The QPSK transmitter may be viewed as <u>two binary PSK generators</u> that work in parallel, each at a <u>bit rate equal to one-half the bit rate</u> of the original binary sequence at the QPSK transmitter input.

Coherent Detection of QPSK Signals



- The QPSK receiver is structured in the form of an *in-phase path* and a *quadrature path*, working in parallel.
- The functional composition of the QPSK receiver is as follows:
- **1.** Pair of correlators, which have a common input $x(t)$.
- The two correlators are supplied with a pair of locally generated orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ which means that the receiver is synchronized with the transmitter.
- The correlator outputs, produced in response to the received signal $x(t)$, are denoted by x_1 and x_2, respectively.
- **2.** Pair of decision devices, which act on the correlator outputs x_1 and x_2 by comparing each one with a zero-threshold; here, it is assumed that the symbols 1 and 0 in the original binary stream at the transmitter input are equally likely.

- If $x_1 > 0$, a decision is made in favor of symbol 1 for the in-phase channel output; on the other hand, if $x_1 < 0$, then a decision is made in favor of symbol 0.
- Similar binary decisions are made for the quadrature channel.
- **3.** Multiplexer, the function of which is to combine the two binary sequences produced by the pair of decision devices.
- The resulting binary sequence so produced provides an *estimate* of the original binary stream at the transmitter input.

Probability of Symbol Error or BER

What is the Probability of **symbol error** for QPSK?

$$P_{MPSK} \approx 2Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

So, P_{sQPSK} = 2Q ($\sqrt{\frac{2E_s}{N_0}}$ sin π/4)= **2Q(** $\boldsymbol{\sqrt{\frac{E_s}{N_0}}}$)

So, P_{eQPSK} = $\frac{1}{\log_2 4}$ x 2Q($\sqrt{\frac{2E_b}{N_0}}$)= **Q(** $\boldsymbol{\sqrt{\frac{2E_b}{N_0}}}$ **)**

Points to Ponder

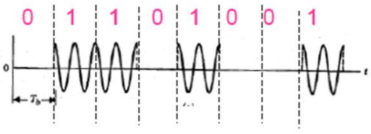
- A QPSK system achieves the same average probability of bit error as a binary PSK system for the same bit rate and the same E_b/N_0, but uses only half the channel bandwidth.
- For the same E_b/N_0 and, therefore, the same average probability of bit error, a QPSK system transmits information at twice the bit rate of a binary PSK system for the same channel bandwidth.
- For a prescribed performance, QPSK uses channel bandwidth better than binary PSK, which explains the preferred use of QPSK over binary PSK in practice.

Binary ASK

- Modulation

"1" $\longrightarrow$ $s_1(t) = \sqrt{\frac{2E}{T_b}}\cos(2\pi f_c t)$

"0" $\longrightarrow$ $s_2(t) = 0 \qquad 0 \le t < T_b$



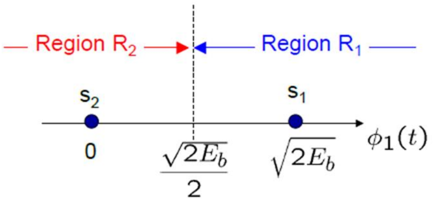
(On-off signaling)

- Average energy per bit

$$E_b = \frac{E+0}{2} \qquad \text{i.e. } E = 2E_b$$

- Decision Region

$$d_{12} = \sqrt{2E_b}$$



Probability of Error for Binary ASK

- Average probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Identical to that of coherent binary FSK

- <u>Exercise:</u> Prove P_e

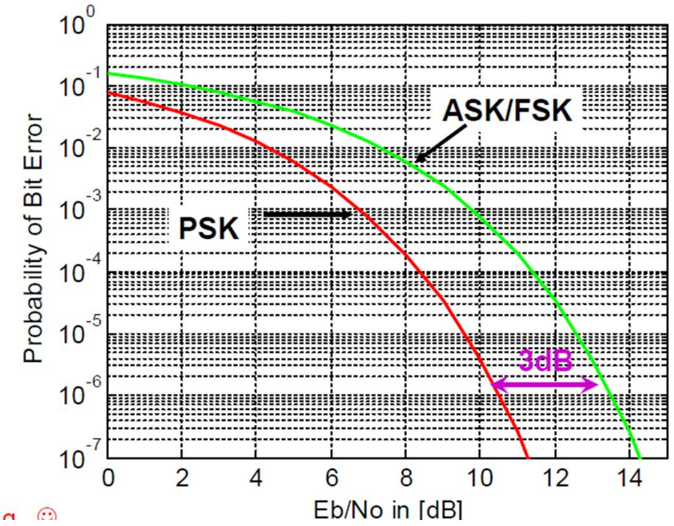
Probability of Error and the Distance Between Signals

BPSK	BFSK	BASK
$d_{1,2} = 2\sqrt{E_b}$	$d_{1,2} = \sqrt{2E_b}$	$d_{1,2} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

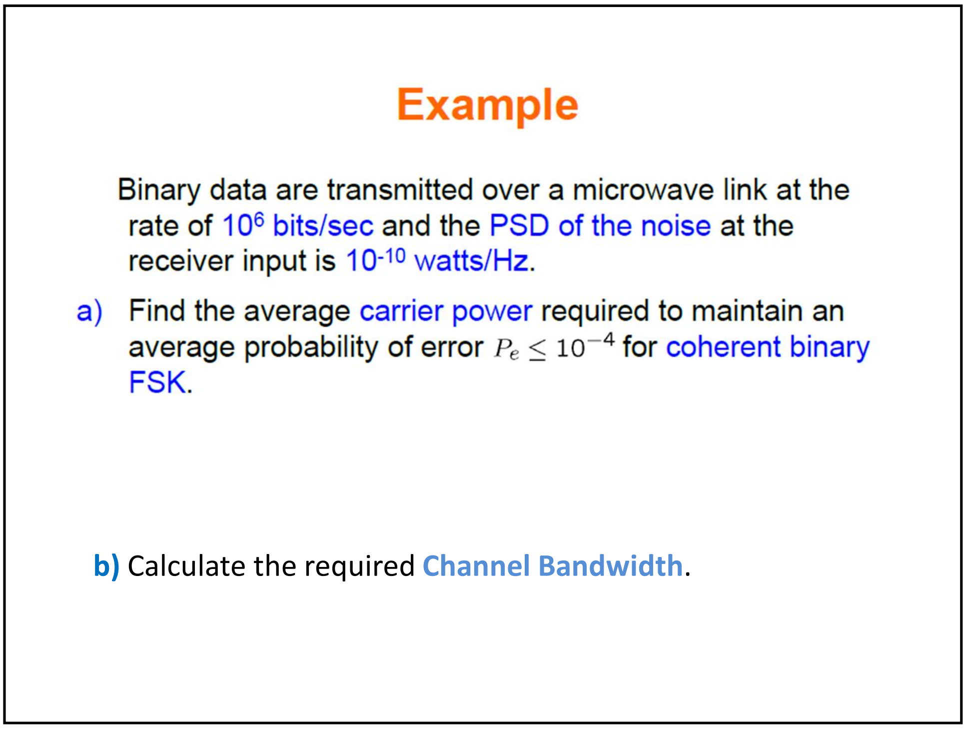
- In general,

$$\boxed{P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)}$$

Probability of Error for BPSK and FSK/ASK



e.g. ☺

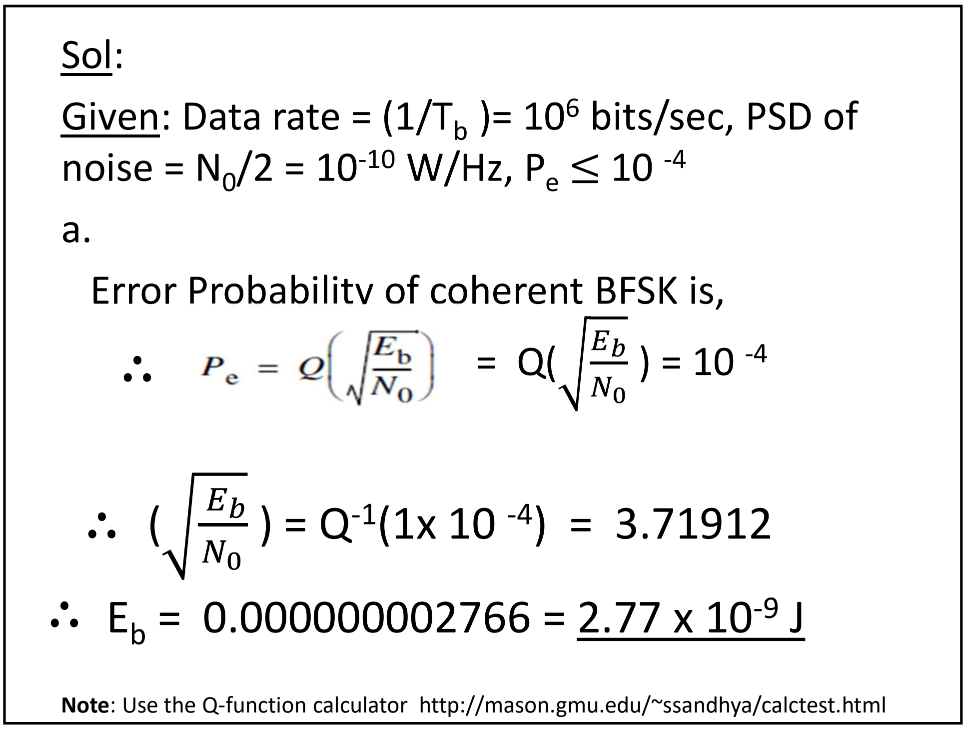


Example

Binary data are transmitted over a microwave link at the rate of 10^6 bits/sec and the PSD of the noise at the receiver input is 10^{-10} watts/Hz.

a) Find the average carrier power required to maintain an average probability of error $P_e \leq 10^{-4}$ for coherent binary FSK.

b) Calculate the required **Channel Bandwidth**.



Sol:

Given: Data rate = $(1/T_b)= 10^6$ bits/sec, PSD of noise = $N_0/2 = 10^{-10}$ W/Hz, $P_e \leq 10^{-4}$

a.

Error Probability of coherent BFSK is,

$$\therefore \quad P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = 10^{-4}$$

$$\therefore \quad \left(\sqrt{\frac{E_b}{N_0}}\right) = Q^{-1}(1\text{x } 10^{-4}) = 3.71912$$

$$\therefore \quad E_b = 0.000000002766 = \underline{2.77 \text{ x } 10^{-9} \text{ J}}$$

Note: Use the Q-function calculator http://mason.gmu.edu/~ssandhya/calctest.html

∴ Power of bit P = E_b/T_b = <u>2.77 x 10^{-9} J</u>/10^{-6}J/s

= <u>0.00277 W</u>

∴ To achieve $P_e \leq 10^{-4}$, average power of carrier must be ≥ <u>0.00277 W</u>

b.

The channel BW is approx. equal to bit rate.

$B_T = 1/T_b = 10^6$ Hz = <u>1 MHz</u>

M-ary signaling scheme

- The no: of possible signals $\boldsymbol{M=2^n}$.
- Symbol duration $\boldsymbol{T=nT_b}$, where $\boldsymbol{T_b}$ is bit duration.
- There are ***M*-ary ASK**, ***M*-ary PSK**, and ***M*-ary FSK**.
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Phase-modulated signals (*M*-ary PSK)

- The *M* signal waveforms are represented as:

$$s_m(t)$$

$$= \mathrm{Re}\left[g(t)e^{j2\pi(m-1)/M}e^{j2\pi f_c t}\right], \quad m = 1,2,\ldots,M, \quad 0 \le t \le T$$

$$= g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right]$$

$$= g(t)\cos\frac{2\pi}{M}(m-1)\cos 2\pi f_c t - g(t)\sin\frac{2\pi}{M}(m-1)\sin 2\pi f_c t$$

- The signal waveforms have equal energy:

$$\varepsilon = \int_0^T s_m^2(t)\,dt = \frac{1}{2}\int_0^T g^2(t)\,dt = \frac{1}{2}\varepsilon_g$$

- The signal waveforms may be represented as a linear combination of **two orthonormal signal waveforms**:

$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}}\, g(t)\cos 2\pi f_c t \text{ and } f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}}\, g(t)\sin 2\pi f_c t$$

- Write down the 2D vector: $\mathbf{s_m}$ =[s_{m1} s_{m2}] ?

- Euclidean distance

$$d_{mn} = \|\mathbf{s}_m - \mathbf{s}_n\| = \sqrt{2E_s\left(1-\cos\frac{2\pi(m-n)}{M}\right)}$$

- The minimum Euclidean distance is

$$d_{\min} = \sqrt{2E_s\left(1-\cos\frac{2\pi}{M}\right)} = 2\sqrt{E_s}\sin\frac{\pi}{M}$$

- $d_{\min}$ plays an important role in determining error performance as
- In the case of PSK modulation, the error probability is dominated by the erroneous selection of either one of the two signal points adjacent to the transmitted signal point.
- Consequently, an approximation to the symbol error probability is

$$P_{MPSK} \approx 2Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

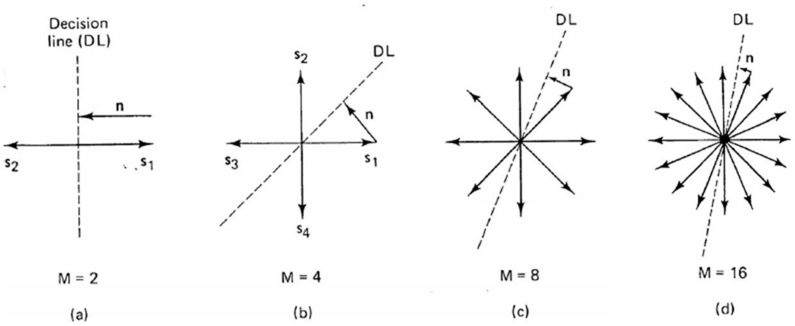
M-ary Modulation

256 QAM

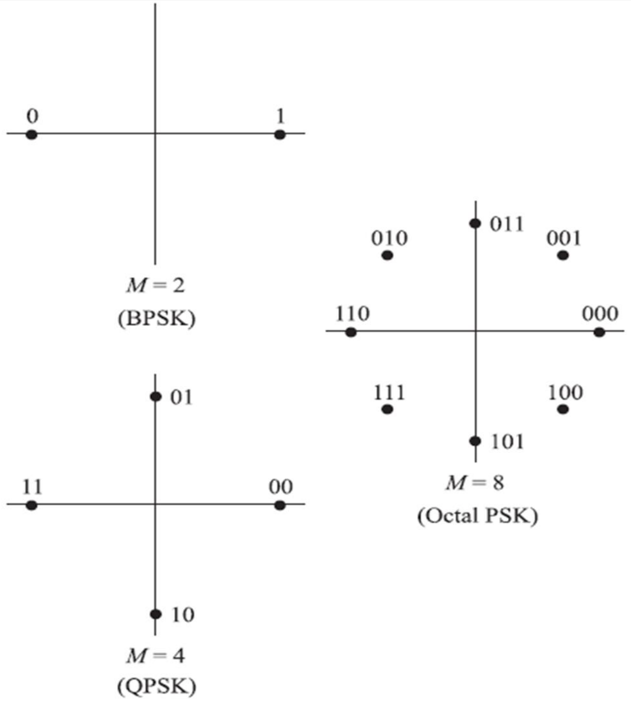
128 QAM

64QAM

MPSK Signal Constellations



BPSK QPSK 8PSK 16PSK



Quadri Phase-Shift Keying

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- Another important goal is the efficient utilization of channel bandwidth.
- Quadri phase shift keying (QPSK) is a bandwidth-conserving modulation scheme, using coherent detection.

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos\left[2\pi \mathrm{f_c} t + (2i-1)\dfrac{\pi}{4}\right], & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2, 3, 4$; E is the transmitted signal energy per symbol, and T is the symbol duration.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[(2i-1)\frac{\pi}{4}\right] \cos(2\pi \mathrm{f_c} t) - \sqrt{\frac{2E}{T}} \sin\left[(2i-1)\frac{\pi}{4}\right] \sin(2\pi \mathrm{f_c} t)$$

- Defined a pair of quadrature carriers:

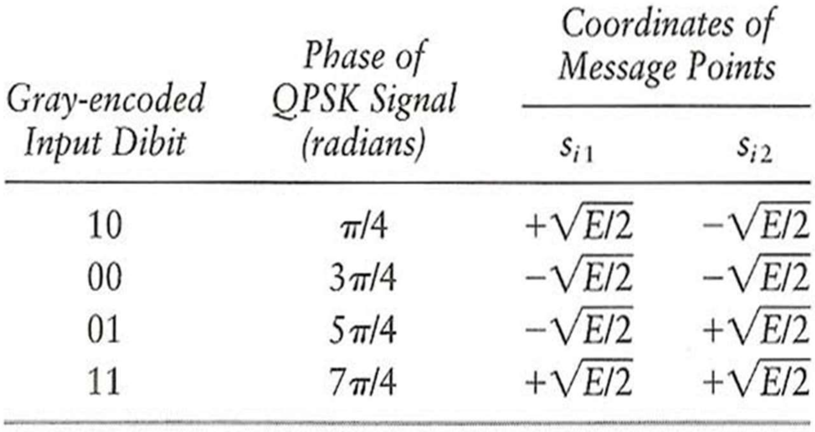
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \qquad 0 \le t \le T$$

- There are **four message points**, and the associated signal vectors are defined by

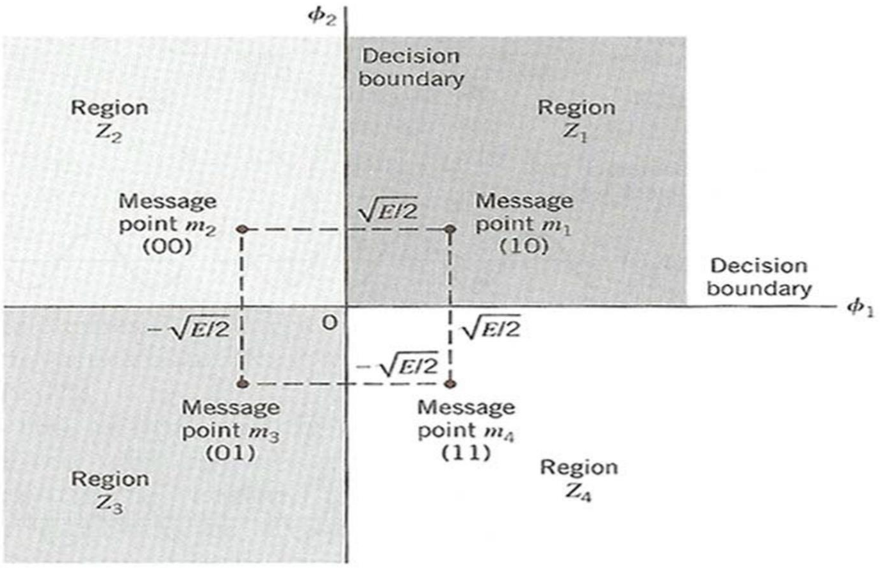
$$s_i(t) = \begin{bmatrix} \sqrt{E} \cos\left((2i-1)\dfrac{\pi}{4}\right) \\ -\sqrt{E} \sin\left((2i-1)\dfrac{\pi}{4}\right) \end{bmatrix}, \qquad i = 1,2,3,4$$

- Each possible value of the phase corresponds to a unique dibit.
- The Gray coding is commonly used.



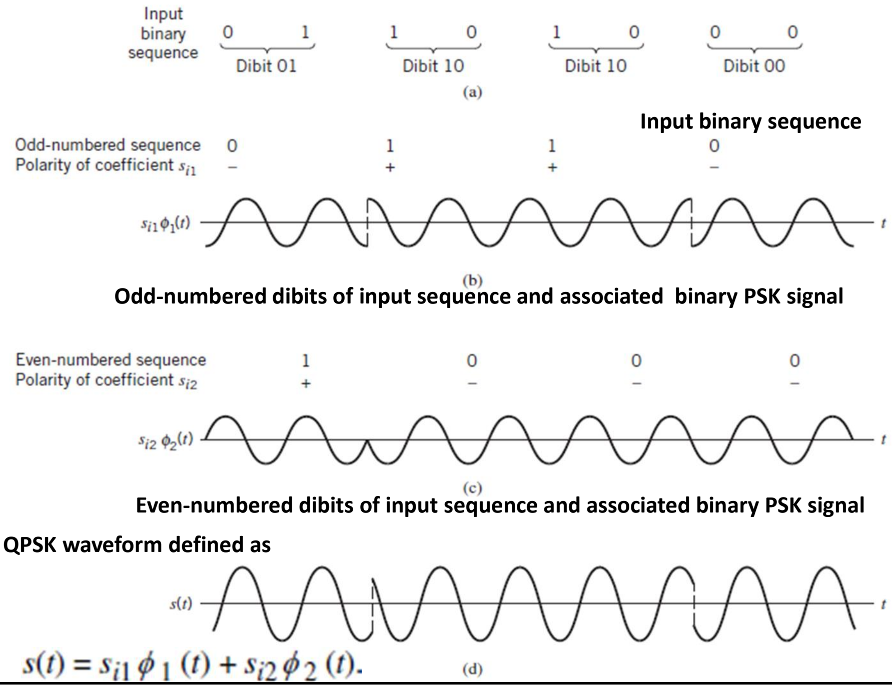
Gray-encoded Input Dibit	Phase of QPSK Signal (radians)	Coordinates of Message Points	
		s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

Signal space diagram of coherent QPSK system



QPSK Waveforms

- Consider the input binary sequence 01101000.
- This sequence is divided into two other sequences, consisting of odd- and even-numbered bits of the input sequence.
- The waveforms representing the two components of the QPSK signal, namely $s_{i1}\phi_1(t)$ and $s_{i2}\phi_2(t)$.
- These two waveforms may individually be viewed as examples of a binary PSK signal.
- Adding them, we get the QPSK waveform.



Input binary sequence

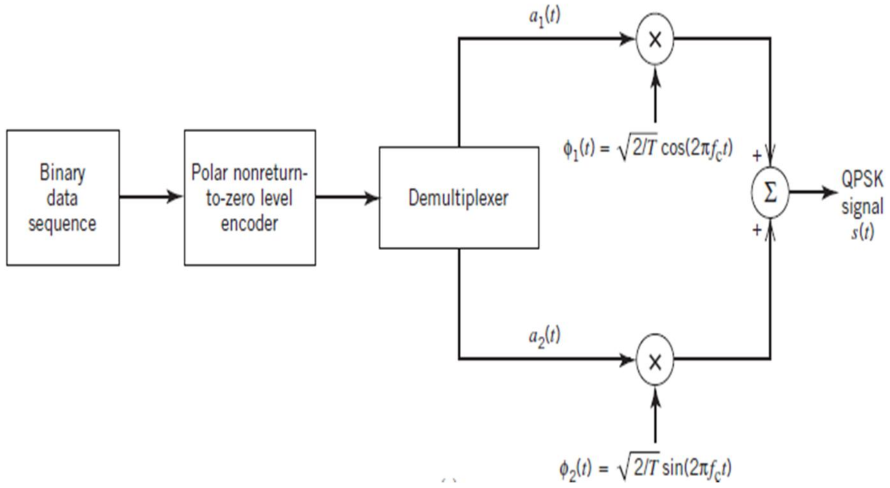
Odd-numbered dibits of input sequence and associated binary PSK signal

Even-numbered dibits of input sequence and associated binary PSK signal

QPSK waveform defined as

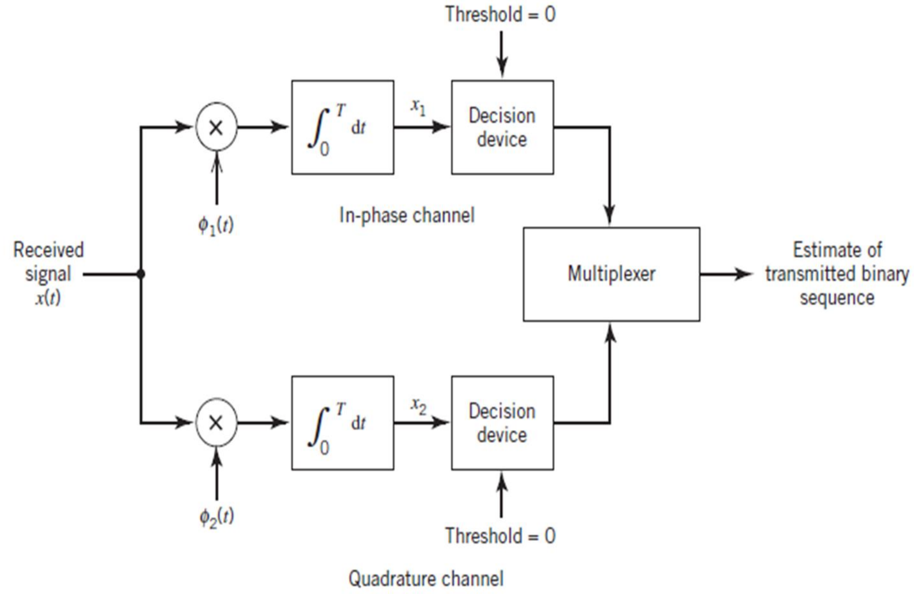
$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t).$$

Generation of QPSK Signals



- A distinguishing feature of the QPSK transmitter is the block labeled Demultiplexer.
- The function of the Demultiplexer is to divide the binary wave produced by the polar NRZ-level encoder into two separate binary waves, one of which represents the odd-numbered dibits in the incoming binary sequence and the other represents the even-numbered dibits.
- The QPSK transmitter may be viewed as two binary PSK generators that work in parallel, each at a bit rate equal to one-half the bit rate of the original binary sequence at the QPSK transmitter input.

Coherent Detection of QPSK Signals



- The QPSK receiver is structured in the form of an *in-phase path* and a *quadrature path*, working in parallel.
- The functional composition of the QPSK receiver is as follows:
- **1.** Pair of correlators, which have a common input $x(t)$.
- The two correlators are supplied with a pair of locally generated orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ which means that the receiver is synchronized with the transmitter.
- The correlator outputs, produced in response to the received signal $x(t)$, are denoted by x_1 and x_2, respectively.
- **2.** Pair of decision devices, which act on the correlator outputs x_1 and x_2 by comparing each one with a zero-threshold; here, it is assumed that the symbols 1 and 0 in the original binary stream at the transmitter input are equally likely.

- If $x_1 > 0$, a decision is made in favor of symbol 1 for the in-phase channel output; on the other hand, if $x_1 < 0$, then a decision is made in favor of symbol 0.
- Similar binary decisions are made for the quadrature channel.
- **3.** Multiplexer, the function of which is to combine the two binary sequences produced by the pair of decision devices.
- The resulting binary sequence so produced provides an *estimate* of the original binary stream at the transmitter input.

Probability of Symbol Error or BER

What is the Probability of **symbol error** for QPSK?

$$P_{MPSK} \approx 2Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

So, $P_{sQPSK} = 2Q\left(\sqrt{\frac{2E_S}{N_0}}\sin \pi/4\right) = \mathbf{2Q\left(\sqrt{\frac{E_s}{N_0}}\right)}$

So, $P_{eQPSK} = \frac{1}{\log_2 4} \times 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = \mathbf{Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$

Points to Ponder

- A QPSK system achieves the same average probability of bit error as a binary PSK system for the same bit rate and the same E_b/N_0, but uses only half the channel bandwidth.
- For the same E_b/N_0 and, therefore, the same average probability of bit error, a QPSK system transmits information at twice the bit rate of a binary PSK system for the same channel bandwidth.
- For a prescribed performance, QPSK uses channel bandwidth better than binary PSK, which explains the preferred use of QPSK over binary PSK in practice.

- We have discussed
  - Coherent modulation schemes, .e.g. BPSK, BFSK, BASK
  - They needs coherent detection, assuming that the receiver is able to detect and track the carrier wave's phase
- In many practical situations, strict phase synchronization is not possible. In these situations, non-coherent reception is required.
- We now consider:
  - Non-coherent detection on binary FSK
  - Differential phase-shift keying (DPSK)

Non-coherent scheme: BFSK

- Consider a binary FSK system, the two signals are

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta_1)$$
$$0 \le t < T_b$$
$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta_2)$$

θ_1, θ_2: unknown random phases with uniform distribution

$$p_{\theta_1}(\theta) = p_{\theta_2}(\theta) = \begin{cases} 1/2\pi & \theta \in [0, 2\pi) \\ 0 & \text{else} \end{cases}$$

Signal Space Representation

- Since

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_1 t + \theta_1\right) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t)\cos(\theta_1) - \sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_1 t)\sin(\theta_1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_2 t + \theta_2\right) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_2 t)\cos(\theta_2) - \sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_2 t)\sin(\theta_2)$$

- Choose four basis functions as

$$\phi_{1c}(t) = \sqrt{2/T_b}\cos(2\pi f_1 t) \quad \phi_{1s}(t) = -\sqrt{2/T_b}\sin(2\pi f_1 t)$$

$$\phi_{2c}(t) = \sqrt{2/T_b}\cos(2\pi f_2 t) \quad \phi_{2s}(t) = \sqrt{2/T_b}\sin(2\pi f_2 t)$$

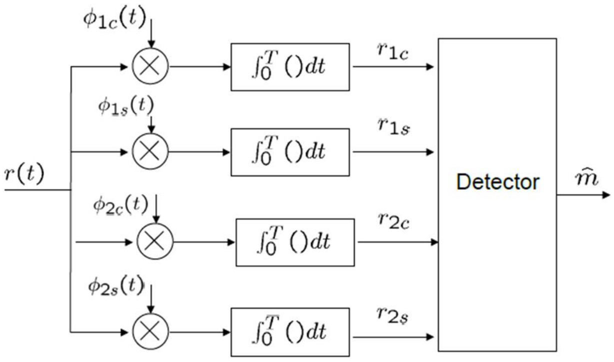
- Signal space representation

$$\vec{s}_1 = [\ \sqrt{E_b}\cos\theta_1 \quad \sqrt{E_b}\sin\theta_1 \quad 0 \quad 0\ ]$$

$$\vec{s}_2 = [\ 0 \quad 0 \quad \sqrt{E_b}\cos\theta_2 \quad \sqrt{E_b}\sin\theta_2\ ]$$

- The vector representation of the received signal

$$\vec{r} = [r_{1c} \quad r_{1s} \quad r_{2c} \quad r_{2s}]$$



Decision Rule for Non-coherent FSK

- ML criterion:

$$f(\vec{r}|\vec{s}_1) \underset{\text{Choose } s_2}{\overset{\text{Choose } s_1}{\gtrless}} f(\vec{r}|\vec{s}_2)$$

- Conditional pdf

$$f(\vec{r}|\vec{s}_1,\theta_1) = \frac{1}{\pi N_0}\exp\left[-\frac{(r_{1c}-\sqrt{E_b}\cos\theta_1)^2+(r_{1s}-\sqrt{E_b}\sin\theta_1)^2}{N_0}\right] \times\frac{1}{\pi N_0}\exp\left[-\frac{r_{2c}^2+r_{2s}^2}{N_0}\right]$$

- Similarly,

$$f(\vec{r}|\vec{s}_2,\theta_2) = \frac{1}{\pi N_0}\exp\left[-\frac{r_{1c}^2+r_{1s}^2}{N_0}\right] \times\frac{1}{\pi N_0}\exp\left[-\frac{(r_{2c}-\sqrt{E_b}\cos\theta_2)^2+(r_{2s}-\sqrt{E_b}\sin\theta_2)^2}{N_0}\right]$$

- For ML decision, we need to evaluate

$$f(\vec{r}|\vec{s}_1) \geq f(\vec{r}|\vec{s}_2)$$

- i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_1,\theta_1)d\theta_1 \geq \frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_2,\theta_2)d\theta_2$$

- Removing the constant terms

$$\left(\frac{1}{\pi N_0}\right)^2 \exp\left[-\frac{{r_{1c}}^2+{r_{1s}}^2+{r_{2c}}^2+{r_{2s}}^2+E}{N_0}\right]$$

- We have the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1$$
$$\geq\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{2c}\cos(\phi_1)+2\sqrt{E}r_{2s}\sin(\phi_1)}{N_0}\right]d\phi_2$$

- By definition

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1 = I_0\left(\frac{2\sqrt{E(r_{1c}^2+r_{1s}^2)}}{N_0}\right)$$

where $I_0(\cdot)$ is a modified Bessel function of the zeroth order

---

Decision Rule (cont'd)

- Thus, the decision rule becomes: choose s_1 if

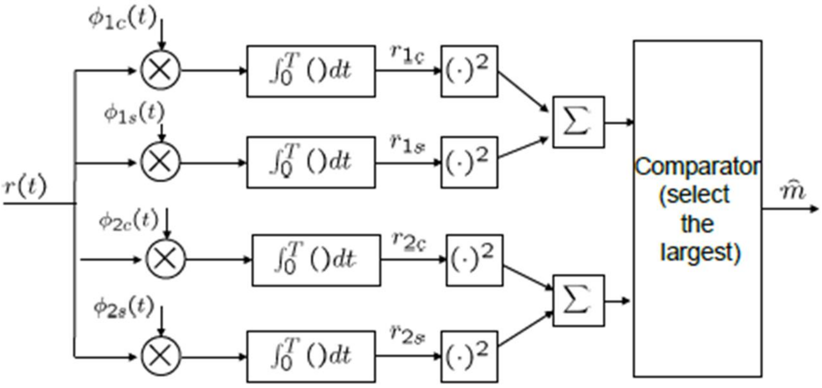
$$I_0\left(\frac{2\sqrt{E(r_{1c}^2+r_{1s}^2)}}{N_0}\right)\geq I_0\left(\frac{2\sqrt{E(r_{2c}^2+r_{2s}^2)}}{N_0}\right)$$

- Noting that this Bessel function is monotonically increasing. Therefore we choose s_1 if

$$\sqrt{r_{1c}^2+r_{1s}^2}\geq\sqrt{r_{2c}^2+r_{2s}^2}$$

- Interpretation: compare the energy in the two frequencies and pick the larger => envelop detector
- Carrier phase is irrelevant in decision making

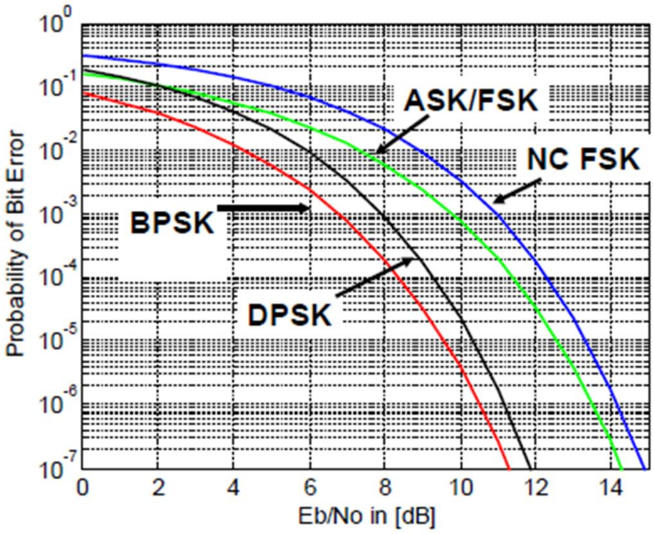
Structure of Non-Coherent Receiver for Binary FSK



- It can be shown that $P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$

(For detailed proof, see Section 10.4.2 in the textbook)

Performance Comparison Between coherent FSK and Non-Coherent FSK



Differential PSK (DPSK)

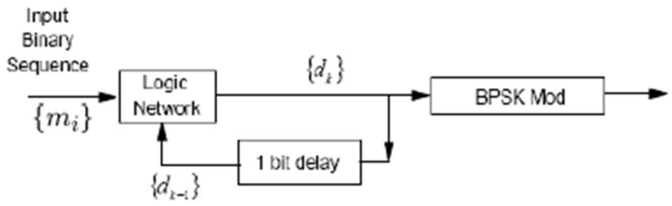
- Non-coherent version of PSK
- Phase synchronization is eliminated using differential encoding
  - Encode the information in phase difference between successive signal transmission. In effect,
  - to send "0", advance the phase of the current signal by 180^0;
  - to send "1", leave the phase unchanged
- Provided that the unknown phase θ contained in the received wave varies slowly (constant over two bit intervals), the phase difference between waveforms received in two successive bit intervals will be independent of θ .

Generation of DPSK signal

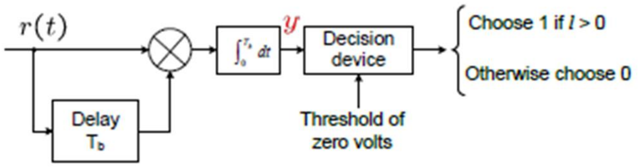
- Generate DPSK signals in two steps
  - Differential encoding of the information binary bits
  - Phase shift keying
- Differential encoding starts with an arbitrary reference bit

Information sequence		1	0	0	1	0	0	1	1	$\{m_i\}$	
Differentially encoded sequence	1 Initial bit	1	0	1	1	0	1	1	1	$\{d_i\}$	$\boxed{d_i = \overline{d_{i-1} \oplus m_i}}$
Transmitted Phase	0	0	π	0	0	π	0	0	0		

DPSK Transmitter Diagram



Differential Detection of DPSK Signals



- Output of integrator (assume noise free)

$$y = \int_0^{T_b} r(t)r(t - T_b)dt = \int_0^{T_b} \cos(w_c t + \psi_k + \theta)\cos(w_c t + \psi_{k-1} + \theta)dt$$

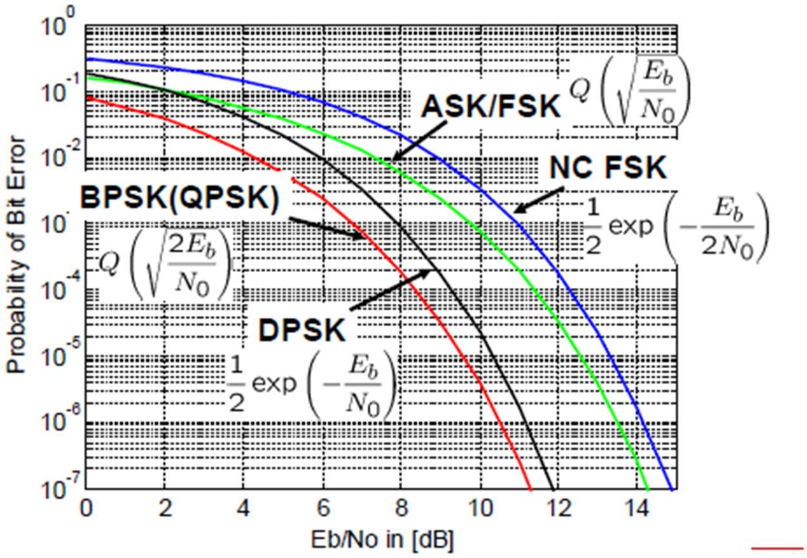
$$\propto \cos(\psi_k - \psi_{k-1})$$

  - The unknown phase θ becomes irrelevant
  - If $\psi_k - \psi_{k-1}$ = 0 (bit 1), then y > 0
  - if $\psi_k - \psi_{k-1}$ =π (bit 0), then y < 0
- Error performance $P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$

Summary of P_e for Different Binary Modulations

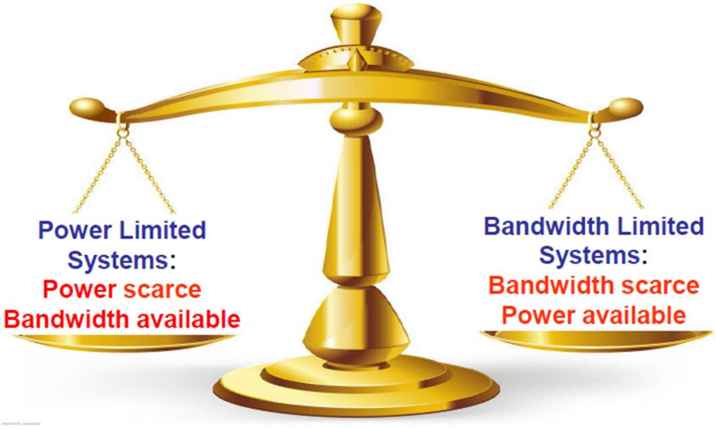
Coherent PSK	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
Coherent ASK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Coherent FSK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Non-Coherent FSK	$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$
DPSK	$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$

P_e Plots for Different Binary Modulations



System Design Tradeoff

Which Modulation to Use ?



Power Limited Systems:
Power scarce
Bandwidth available

Bandwidth Limited Systems:
Bandwidth scarce
Power available

Practical Applications

- BPSK:
    - WLAN IEEE802.11b (1 Mbps)
- QPSK:
    - WLAN IEEE802.11b (2 Mbps, 5.5 Mbps, 11 Mbps)
    - 3G WDMA
    - DVB-T (with OFDM)
- QAM
    - Telephone modem (16QAM)
    - Downstream of Cable modem (64QAM, 256QAM)
    - WLAN IEEE802.11a/g (16QAM for 24Mbps, 36Mbps; 64QAM for 38Mbps and 54 Mbps)
    - LTE Cellular Systems
- FSK:
    - Cordless telephone
    - Paging system

Spread Spectrum Systems
Pseudo–Noise Sequences

EC302 DC Module V

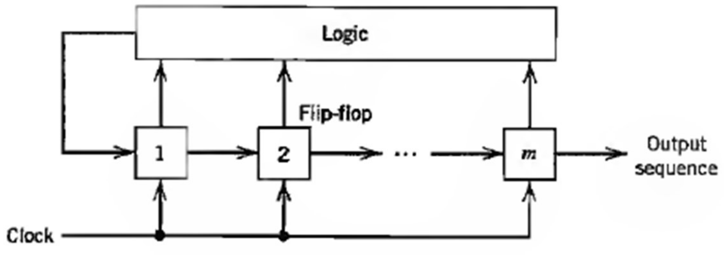
Introduction

- A major issue of concern in the study of digital communications is that of providing for the efficient use of bandwidth and power.
- There are situations where it is necessary to sacrifice this efficiency in order to meet certain other design objectives.
- For example, the system may be required to provide a form of secure communication in a 'hostile' environment such that the transmitted signal is not easily detected or recognized by unwanted listeners.
- This requirement is provided to by a class of signaling techniques known collectively as <u>Spread-Spectrum Modulation</u>.

- The primary advantage of a Spread-Spectrum Communication system is its ability to reject Interference whether it be :
  - the unintentional interference by another user simultaneously attempting to transmit through the channel, or
  - the intentional interference by a hostile transmitter attempting to jam the transmission.
- Spread Spectrum is a means of transmission in which the data sequence occupies a bandwidth in excess of the minimum bandwidth necessary to send it.
- The Spectrum Spreading is accomplished before transmission through the use of a code that is independent of the data sequence.
- The same code is used in the receiver(operating in synchronism with the transmitter) to despread the received signal so that the original data sequence may be recovered.

Pseudo-Noise Sequences

- A pseudo-noise (PN) sequence is a periodic binary sequence with a noiselike waveform that is usually generated by means of a Feedback Shift Register, as shown below.



- A feedback shift register consists of an ordinary shift register made up of m flip-flops (two-state memory stages) and a logic circuit that are interconnected to form a multiloop feedback circuit.
- The flip-flops in the shift register are regulated by a single timing clock.
- At each pulse (tick) of the clock, the state of each flip-flop is shifted to the next one down the line.
- With each clock pulse the logic circuit computes a Boolean function of the states of the flip-flops.
- The result is then fed back as the input to the first flip-flop, thereby preventing the shift register from emptying.
- The PN sequence so generated is determined by the length m of the shift register, its initial state, and the feedback logic.

- Let $s_j(k)$ denote the state of the j^{th} flip-flop after the k^{th} clock pulse;
- This state is represented by symbol 0 or 1.
- The state of the shift register after the k^{th} clock pulse is then defined by the set $\{s_1(k), s_2(k), .., s_m(k)\}$, where $k \geq 0$.
- For the initial state, k is zero.
- From the definition of a shift register,

$$s_j(k+1) = s_{j-1}(k), \qquad \begin{cases} k \geq 0 \\ 1 \leq j \leq m \end{cases} \qquad \text{.......(1)}$$

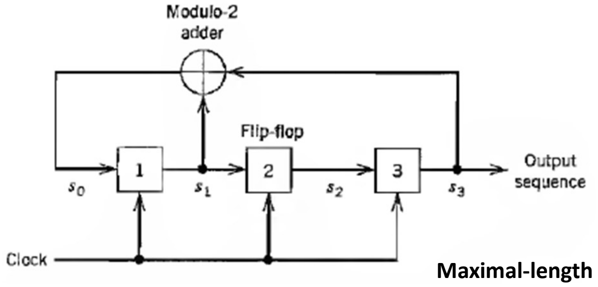
- where $s_0(k)$ is the input applied to the first flip-flop after the k^{th} clock pulse.

- $s_0(k)$ is a Boolean function of the individual states $s_1(k)$, $s_2(k)$, .., $s_m(k)$.
- For a specified length *m*, this Boolean function uniquely determines the subsequent sequence of states and therefore the PN sequence produced at the output of the final flip-flop in the shift register.
- With a total number of *m* flip-flops, the number of possible states of the shift register is at most 2^m.
- It follows therefore that the PN sequence generated by a feedback shift register must eventually become periodic with a period of at most 2^m.

- A feedback shift register is said to be linear when the feedback logic consists entirely of modulo-2 adders.
- In such a case, the zero state (e.g., the state for which all the flip-flops are in state 0) is not permitted. Why?
- For a zero state, the input $s_0(k)$ produced by the feedback logic would be 0, the shift register would then continue to remain in the zero state, and the output would therefore consist entirely of 0s.
- Consequently, the period of a PN sequence produced by a linear feedback shift register with *m* flip-flops cannot exceed $2^m — 1$.
- When the period is exactly $2^m — 1$, the PN sequence is called a maximal-length-sequence or m-sequence.

Example-1

- Consider the linear feedback shift register shown, involving three flip-flops.
- The input s_0 applied to the first flip-flop is equal to the modulo-2 sum of s_1 and s_3.
- It is assumed that the initial state of the shift register is 100 (reading the contents of the three flip-flops from left to right).
- Then, the succession of states will be as follows:
- 100, 110, 111, 011, 101, 010, 001, 100,....



Maximal-length sequence generator for m=3

- The output sequence (the last position of each state of the shift register) is therefore,
- 00111010 . ..
- which repeats itself with period $2^3 - 1 = 7$.
- Note that the choice of 100 as the initial state is arbitrary.
- Any of the other six permissible states could serve equally well as an initial state.
- The resulting output sequence would then simply experience a cyclic shift.

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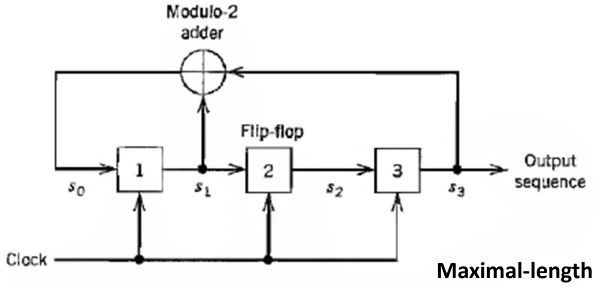

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- The resulting output sequence would then simply experience a cyclic shift.

Properties of Maximal-Length Sequences

- Maximal-length sequences have many of the properties possessed by a truly random binary sequence.
- A random binary sequence is a sequence in which the presence of binary symbol 1 or 0 is equally probable.
- Some properties of maximal-length sequences are as follows:

1. In each period of a maximal-length sequence, the number of 1s is always one more than the number of 0s.

This property is called the balance property.

2. Among the runs of 1s and of 0s in each period of a maximal length sequence, one- half the runs of each kind are of length one, one-fourth are of length two, one-eighth are of length three, and so on as long as these fractions represent meaningful numbers of runs.

- This property is called the run property.
- A "run" means a subsequence of identical symbols (1s or 0s) within one period of the sequence.
- The length of this subsequence is the length of the run.
- For a maximal-length sequence generated by a linear feedback shift register of length m, the total number of runs is $(N + 1)/2$, where $N = 2^m - 1$.

3. The autocorrelation function of a maximal-length sequence is periodic and binary- valued. This property is called the correlation property.

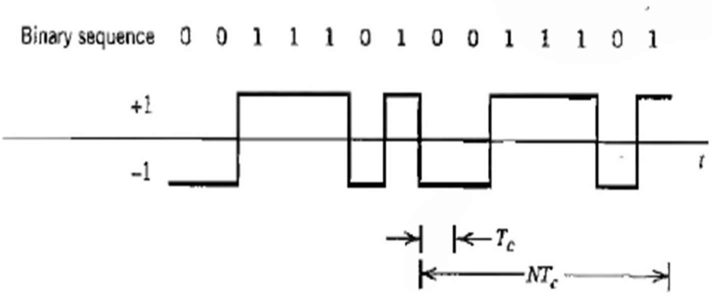
- The period of a maximum-length sequence is defined by,

$$N = 2^m - 1 \quad …….(2)$$

- where *m* is the length of the shift register.
- Let binary symbols 0 and 1 of the sequence be denoted by the levels -1 and +1, respectively.
- Let c(t) denote the resulting waveform of the maximal-length sequence for N= 7.
- The period of the waveform c(t) is,

$$T_b = NT_c \quad …….(3)$$

- where T_c is the duration assigned to symbol 1 or 0 in the maximal-length sequence.



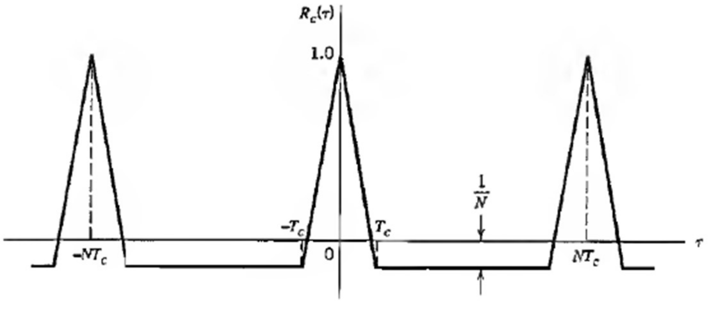
Waveform of maximal-length sequence for length m = 3 or period N = 7

- By definition, the autocorrelation function of a periodic signal c(t) of period T_b is,

$$R_c(\tau) = \frac{1}{T_b}\int_{-T_b/2}^{T_b/2} c(t)c(t-\tau)\,dt \qquad \text{.......(4)}$$

- where the lag τ lies in the interval $(-T_b/2, T_b/2)$
- Applying this formula to a maximal-length sequence represented by c(t),

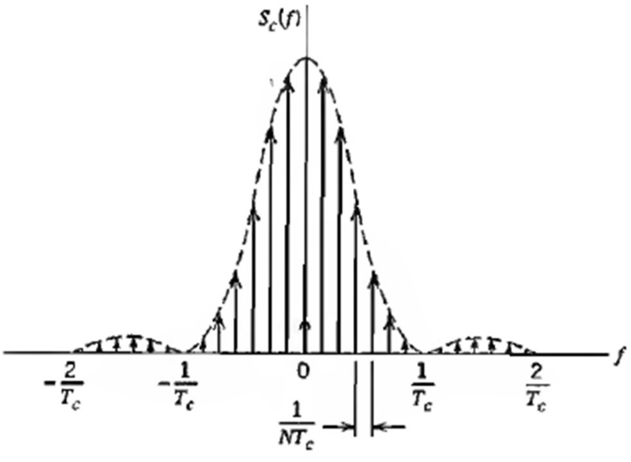
$$R_c(\tau) = \begin{cases} 1 - \dfrac{N+1}{NT_c}|\tau|, & |\tau| \le T_c \\ -\dfrac{1}{N}, & \text{for the remainder of the period} \end{cases} \qquad \text{.......(5)}$$



Plot of Autocorrelation function for the case of m = 3 or N = 7

- From Fourier transform theory, periodicity in the time domain is transformed into uniform sampling in the frequency domain.
- This interplay between the time and frequency domains is borne out by the Power Spectral Density (PSD) of the maximal-length wave c(t).
- Specifically, taking the Fourier transform of eqn (5), the sampled spectrum is got,

$$S_c(f) = \frac{1}{N^2}\,\delta(f) + \frac{1+N}{N^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \operatorname{sinc}^2\left(\frac{n}{N}\right) \delta\left(f - \frac{n}{NT_c}\right) \quad \text{.......(6)}$$

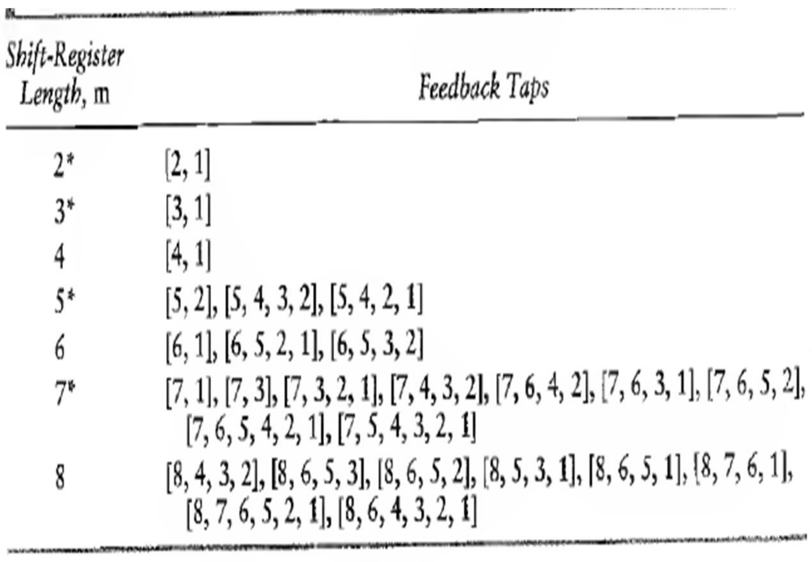


Power spectral density for the case of m = 3 or N = 7

- As the shift-register length m, or equivalently, the period N of the maximal-length sequence is increased, the maximal-length sequence becomes increasingly similar to the random binary sequence.
- Indeed, in the limit, the two sequences become identical when N is made infinitely large.
- However, the price paid for making N large is an increasing storage requirement, which imposes a practical limit on how large N can actually be made.

Choosing a Maximal-Length Sequence

- How to find the feedback logic for a desired period N?
- The task of finding the required feedback logic is made particularly easy by virtue of the extensive tables of the necessary feedback connections for varying shift-register lengths that have been compiled in the literature.
- Table -1, presents the sets of maximal (feedback) taps pertaining to shift-register lengths, $m = 2, 3,..., 8$.
- Note that as *m* increases, the number of alternative schemes (codes) is enlarged.
- Also, for every set of feedback connections shown in this table, there is an “image” set that generates an identical maximal-length code, reversed in time sequence.
- The particular sets identified with an asterisk in Table-1 correspond to Mersenne prime length sequences, for which the period N is a prime number.

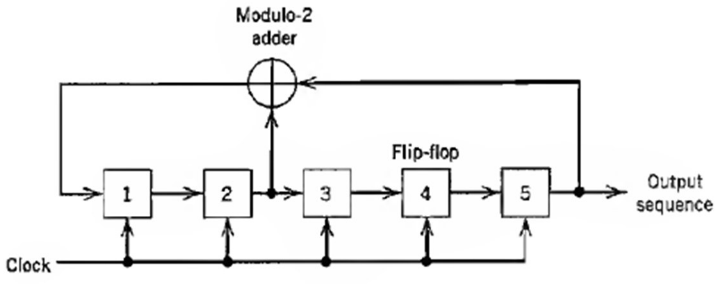


Shift-Register Length, m	*Feedback Taps*
2*	[2, 1]
3*	[3, 1]
4	[4, 1]
5*	[5, 2], [5, 4, 3, 2], [5, 4, 2, 1]
6	[6, 1], [6, 5, 2, 1], [6, 5, 3, 2]
7*	[7, 1], [7, 3], [7, 3, 2, 1], [7, 4, 3, 2], [7, 6, 4, 2], [7, 6, 3, 1], [7, 6, 5, 2], [7, 6, 5, 4, 2, 1], [7, 5, 4, 3, 2, 1]
8	[8, 4, 3, 2], [8, 6, 5, 3], [8, 6, 5, 2], [8, 5, 3, 1], [8, 6, 5, 1], [8, 7, 6, 1], [8, 7, 6, 5, 2, 1], [8, 6, 4, 3, 2, 1]

Table – 1 : Maximal-length sequences of shift-register lengths 2—8

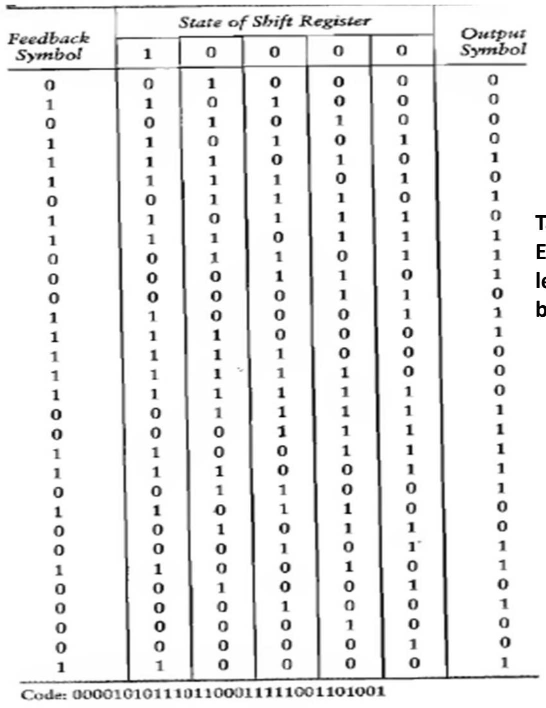
Example -2

- Consider a maximal-length sequence requiring the use of a linear feedback-shift register of length m = 5.
- For feedback taps, select set [5, 2] from Table -1.



Configuration of the code generator
Feedback connections [5, 2]

- Assuming that the initial state is 10000, the evolution of one period of the maximal-length sequence generated by this scheme is shown in Table 7.2a, where the generator returns to the initial 10000 after 31 iterations;
- that is, the period is 31, which agrees with the value obtained from eqn (2).

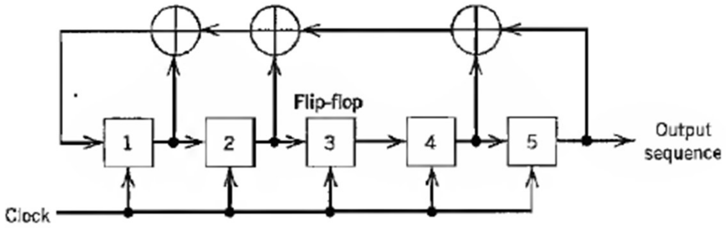


Feedback Symbol	State of Shift Register					Output Symbol
	1	0	0	0	0	
0	0	1	0	0	0	0
1	1	0	1	0	0	0
0	0	1	0	1	0	0
1	1	0	1	0	1	0
1	1	1	0	1	0	1
1	1	1	1	0	1	0
0	0	1	1	1	0	1
1	1	0	1	1	1	0
1	1	1	0	1	1	1
0	0	1	1	0	1	1
0	0	0	1	1	0	1
0	0	0	0	1	1	0
1	1	0	0	0	1	1
1	1	1	0	0	0	1
1	1	1	1	0	0	0
1	1	1	1	1	0	0
1	1	1	1	1	1	0
0	0	1	1	1	1	1
0	0	0	1	1	1	1
1	1	0	0	1	1	1
1	1	1	0	0	1	1
0	0	1	1	0	0	1
1	1	0	1	1	0	0
0	0	1	0	1	1	0
0	0	0	1	0	1	1
1	1	0	0	1	0	1
0	0	1	0	0	1	0
0	0	0	1	0	0	1
0	0	0	0	1	0	0
0	0	0	0	0	1	0
1	1	0	0	0	0	1

Code: 0000101011101100011111001101001

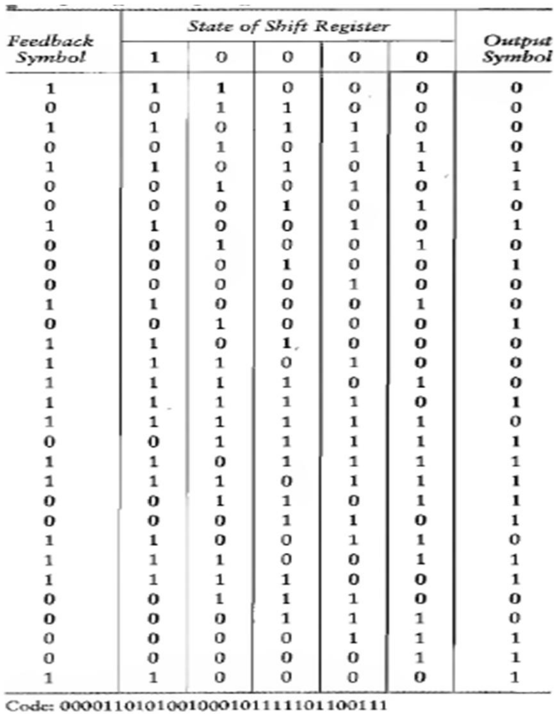
Table-2a:
Evolution of the maximal-length sequence generated by the feedback- shift register

- Suppose next select another set of feedback taps front Table -1, namely, [5,4,2,1].



Feedback connections [5, 4, 2, 1]

- For the initial state 10000, now find that the evolution of the maximal-length sequence is as shown in Table -2b.
- Here again, the generator returns to the initial state 10000 after 31 iterations, and so it should.
- But the maximal-length sequence generated is different from that shown in Table -2a.
- Clearly, the first code generator has an advantage over the second, as it requires fewer feedback connections.



Feedback Symbol	*State of Shift Register*					*Output Symbol*
	1	0	0	0	0	
1	1	1	0	0	0	0
0	0	1	1	0	0	0
1	1	0	1	1	0	0
0	0	1	0	1	1	0
1	1	0	1	0	1	1
0	0	1	0	1	0	1
0	0	0	1	0	1	0
1	1	0	0	1	0	1
0	0	1	0	0	1	0
0	0	0	1	0	0	1
0	0	0	0	1	0	0
1	1	0	0	0	1	0
0	0	1	0	0	0	1
1	1	0	1	0	0	0
1	1	1	0	1	0	0
1	1	1	1	0	1	0
1	1	1	1	1	0	1
1	1	1	1	1	1	0
0	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	0	1	1	1
0	0	1	1	0	1	1
0	0	0	1	1	0	1
1	1	0	0	1	1	0
1	1	1	0	0	1	1
1	1	1	1	0	0	1
0	0	1	1	1	0	0
0	0	0	1	1	1	0
0	0	0	0	1	1	1
0	0	0	0	0	1	1
1	1	0	0	0	0	1

Code: 0000110101001000101111101100111

Table-2b: Evolution of the maximal- length sequence generated by the feedback- shift register

Maximal-Length and Gold Codes

- Code-division multiplexing (CDM) provides an alternative to the traditional methods of frequency-division multiplexing (FDM) and time-division multiplexing (TDM).
- It does not require the bandwidth allocation of FDM nor the time synchronization needed in TDM.
- Rather, users of a common channel are permitted access to the channel through the assignment of a “spreading code” to each individual user under the umbrella of spread-spectrum modulation.
- In an ideal CDM system, the cross-correlation between any two users of the system is zero.
- For this ideal condition to be realized, the cross-correlation function between the spreading codes assigned to any two users of the system be zero for all cyclic shifts.
- Unfortunately, ordinary PN sequences do not satisfy this requirement because of their relatively poor cross-correlation properties.
- As a remedy for this shortcoming of ordinary PN sequences, use a special class of PN sequences called Gold sequences (codes) .

Gold's Theorem

- Let $g_1(X)$ and $g_2(X)$ be a preferred pair of primitive polynomials of degree *n* whose corresponding shift registers generate maximal-length sequences of period $2^n - 1$ and whose cross-correlation function has a magnitude less than or equal to,

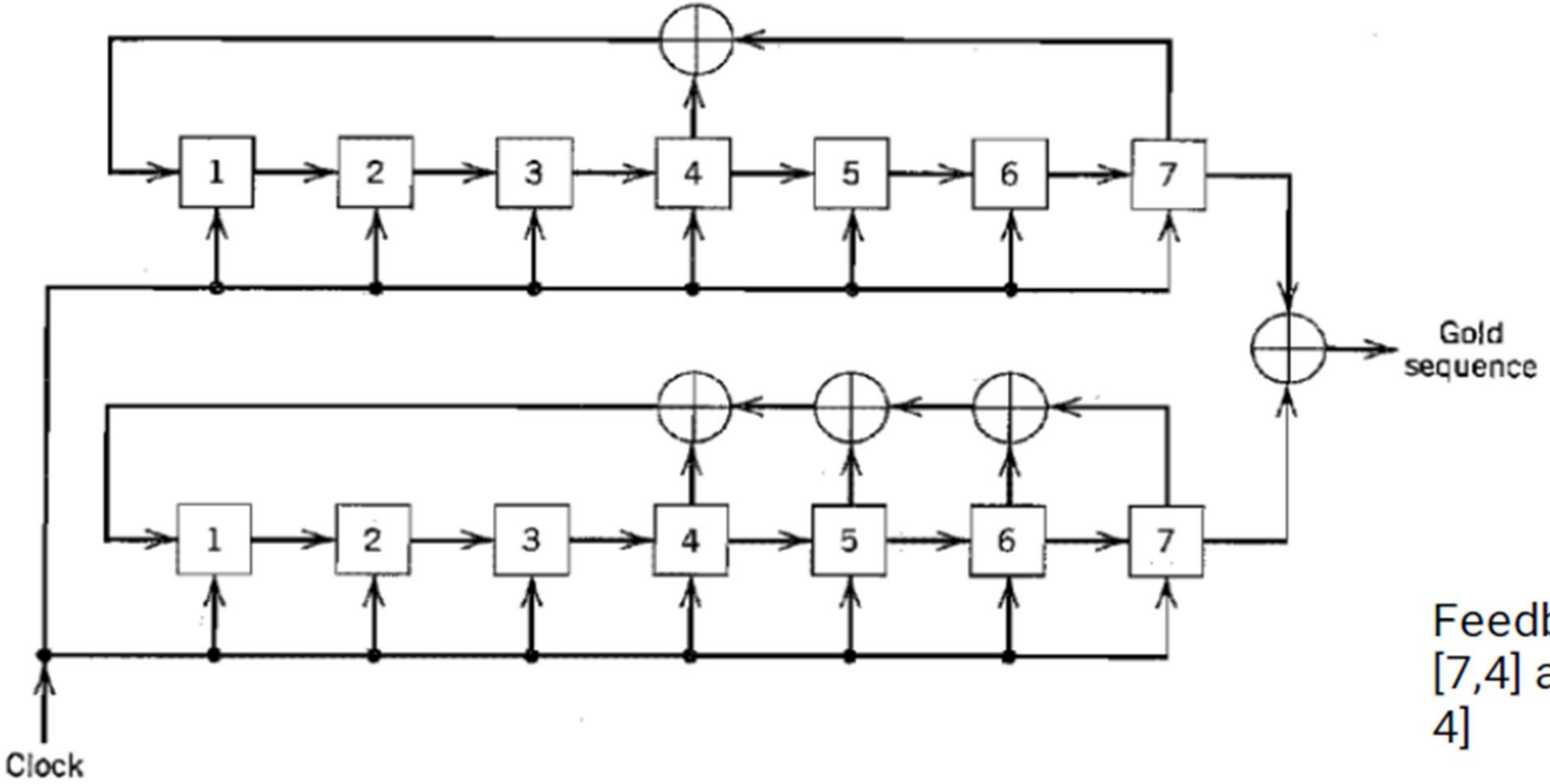
$$\begin{cases} 2^{(n+1)/2} + 1 & \text{for } n \text{ odd} \\ 2^{(n+2)/2} + 1 & \text{for } n \text{ even and } n \neq 0 \bmod 4 \end{cases} \qquad \text{.......(7)}$$

- Then the shift register corresponding to the product polynomial $g_1(X).g_2(X)$ will generate 2^n+1 different sequences, with each sequence having a period of $2^n - 1$, and the cross-correlation between any pair of such sequences satisfying the above condition.

- To understand **Gold's theorem**, need to define what is a primitive polynomial?
- Consider a polynomial g(X) defined over a binary field (i.e,, a finite set of two elements, 0 and 1, which is governed by the rules of binary arithmetic).
- The polynomial g(X) is said to be an irreducible polynomial if it cannot be factored using any polynomials from the binary field.
- An irreducible polynomial g(X) of degree *m* is said to be a primitive polynomial if the smallest integer m for which the polynomial g(X) divides the factor $X^n + 1$ is $n = 2^m$-1.

Gold codes

- In an ideal CDM (system), the cross correlation between any two users of the system is zero.
- For this ideal condition, we require that the cross correlation function between the spreading codes assigned to any two users of the system be zero for all cyclic shifts.
- Unfortunately, ordinary PN sequences do not satisfy this requirement .
- As a remedy for this shortcoming of ordinary PN sequences, we may use special class of PN sequences called Gold Codes.



Feedback taps [7,4] and [7,6,5, 4]

FIGURE 7.14 Generator for a Gold sequence of period $2^7 - 1 = 127$.

- To uniquely define a Gold code:
  1. State characteristic polynomial for two LFSRs.
  2. State seed(initial stage) for the second LFSR.
  3. Always use seed of 000…01 for the first LFSR.
- Example: GC(1+x2+ x3+ x4 + x5, 1+x3+x5, 00011)
- Gold code is formed by XOR ing MLSR (Maximum Length shift Registers) generated by different taps.
- Maximum cross correlation magnitude of Gold codes is $2^{(N+1)/2}+1$ for N odd.
- For N even it is $2^{(N+2)/2}+1$
- For N=7 , cross correlation is less than or equal to 17.

Synchronization

EC302 DC Module V Part-2

- The coherent reception of a digitally modulated signal requires that the receiver be synchronous to the transmitter.
- Two sequences of events (representing a transmitter and a receiver) occur simultaneously.
- The process of making a situation synchronous, and maintaining it in this condition, is called synchronization.
- When coherent detection is used, knowledge of both the frequency and phase of the carrier is necessary.
- The estimation of carrier phase and frequency is called carrier recovery or carrier synchronization.
- To perform demodulation, the receiver has to know the starting and finishing times of the individual symbols.
- The estimation of these times is called clock recovery or symbol synchronization.

1. Carrier synchronization
2. Symbol or clock synchronization
3. Frame synchronization

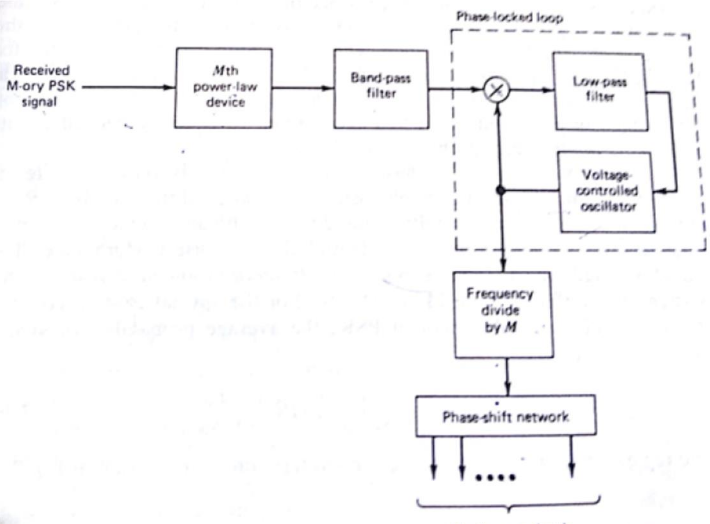
Carrier Synchronization

- The most straight forward method is to modulate the data bearing signal onto a carrier in such a way that the power spectrum of the modulated signal contains a discrete component at the carrier frequency.
- Then a narrow band phase locked loop can be used to track this component thereby providing the desired reference signal at the receiver.
- Phase locked loop consists of a voltage controlled oscillator(VCO), a loop filter and a multiplier that are connected together in the form of a negative feedback system.
- The disadvantage of such an approach is that since the residual component does not convey any information other than the frequency and phase of the carrier, its transmission represents a waste of power.

- Carrier synchronization done by two methods:
  1. Mth power loop
  2. Costas loop

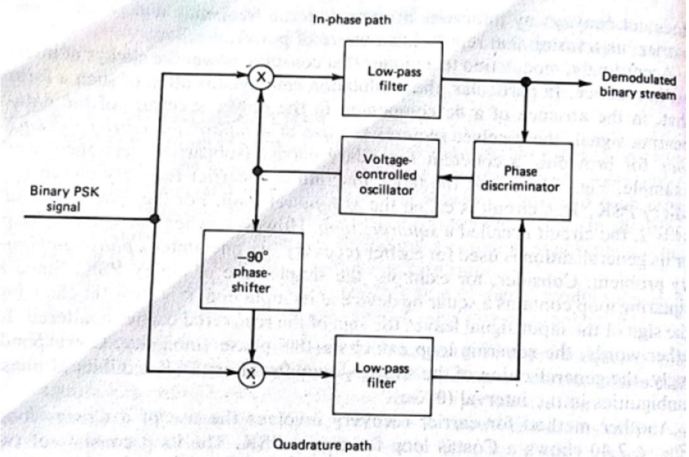
1. Mth power loop

- Block diagram shows a carrier recovery circuit for M ary PSK.
- This circuit is called Mth power loop. For the special case of M = 2 it is called squaring loop.
- Input signal first raised to Mth power.
- Then signal passed through the bandpass filter tuned to the carrier frequency f_c.
- PLL tracks the carrier frequency.
- Output of VCO is the carrier frequency.
- Output of VCO is divided by M.
- Phase shift network then separates the M reference signal for the M correlation receiver.
- When M=2, input signal is squared. Therefore sign of recovered carrier is always independent of sign of input signal. Therefore there can be 180° error in the output.



Mth power loop.

2. Costas Loop



Costas loop.

- Another method for carrier recovery involves the use of a *Costas Loop.*
- The block diagram shows the costas loop for binary PSK.
- The loop consists of two paths, one referred to as in phase and the other referred to as quadrature.
- Both are coupled together via a common voltage controlled oscillator(VCO) to form a negative feedback system.
- When synchronization is attained, the demodulated data waveform appears at the output of the in phase path, and the corresponding output of the quadrature path is zero under ideal conditions.

- Costas loop also exhibits the same phase ambiguity problem as the squaring loop.
- Moreover the Costas loop is equivalent to the squaring loop in terms of noise performance, provided that the two low pass filters in the two paths of the Costas loop are the low pass equivalent of the bandpass filter in the squaring loop.
- The Costas loop may be generalized for M ary PSK, in which case it exhibits M ambiguities in the interval (0,2π)

Disadvantage of Costas loop

- Compared to the M$_{th}$ order loop, the M th order Costas loop has a practical disadvantage in that the amount of circuitry needed for its implementation becomes prohibitive for large M.
- One method of resolving the phase ambiguity problem is to exploit differential encoding.
  - Specifically the incoming data sequence is differentially encoded before modulation at the transmitter, and differentially decoded after detection at the receiver. It is called the *coherent detection of differentially encoded M ary PSK*

Symbol Synchronization(clock recovery)

- Symbol synchronization can be processed along with carrier recovery.
- In one approach , the symbol synchronization problem is solved by transmitting a clock along with the data bearing signal, in multiplexed form.
- Then at the receiver, the clock is extracted by appropriate filtering of the modulated waveforms.
- Such an approach minimizes the time required for carrier/ clock recovery.
- A disadvantage of this method is that a fraction of the transmitted power is allocated to the transmission of the clock.

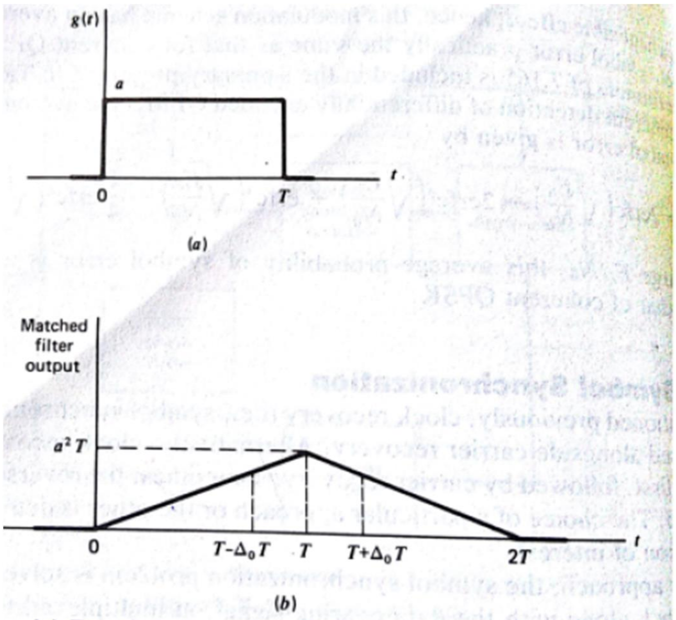
- In another approach, first to use a non coherent detector to extract the clock.
- The clock timing is usually much more stable than carrier phase. Then the carrier is recovered by processing the non coherent detector output in each clocked interval.

- In yet another approach, when clock recovery follows carrier recovery, the clock is extracted by processing the demodulated baseband waveforms, thereby avoiding any wastage of transmitted power.

- Consider first a rectangular pulse defined by

$$g(t) = \begin{cases} a & 0 \leq t \leq T \\ 0 & otherwise \end{cases}$$

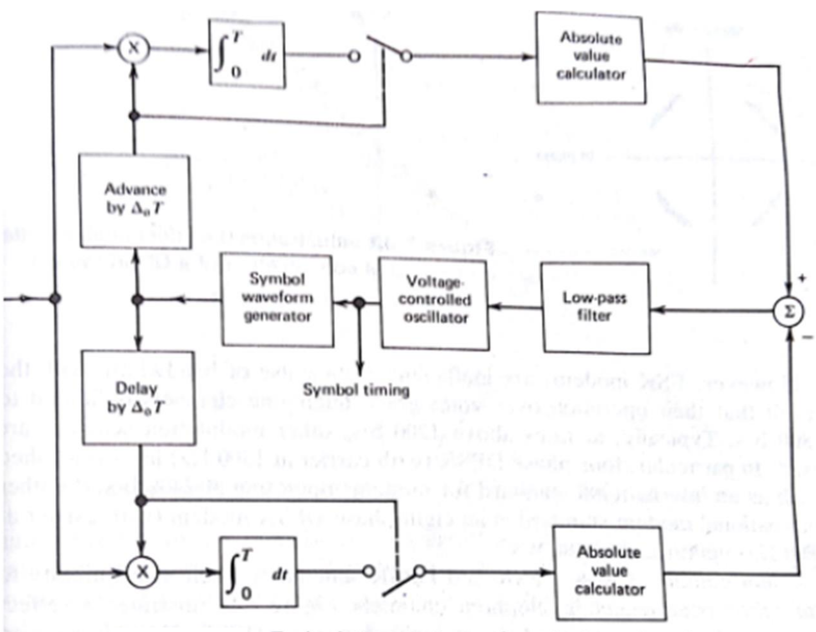
- The output of a filter matched to the pulse g(t) is shown in figure.
- The matched filter output attains its peak value at time t = T, and that it is symmetric about this point.
- Clearly the proper time to sample the matched filter output is at t =T.



(*a*) Rectangular pulse, *g*(*t*). (*b*) Output of filter matched

- Consider the matched filter output sampled early at $t = T - \Delta_0 T$ or late at $t = T + \Delta_0 T$.

- Then the absolute values of the two samples so obtained will (on the average in the presence of additive noise) be equal, and smaller than the peak value at $t = T$.
- Error signal is the difference between absolute values of the two samples, ie zero.
- The proper sampling time is mid point between $t = T - \Delta_0 T$ and $t = T + \Delta_0 T$.
- This special condition may be viewed as equilibrium point, in that if we deviate from it the error signal becomes non zero.

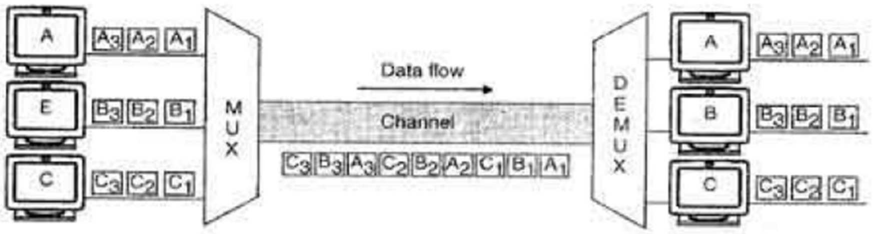


Early–late gate type of symbol synchronizer.

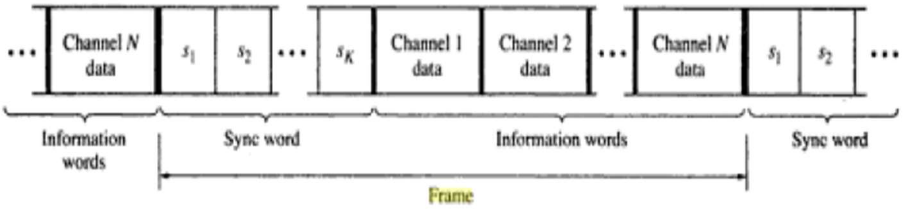
- It is called an early – late gate symbol synchronizer of the absolute value type.
- Correlators are used in place of equivalent matched filters.
- Both correlators integrate over a full symbol interval T, with one starting Δ_0 T early relative to the transmission time estimate and other starting Δ_0 T late.
- An error signal e(kT) is generated by taking the difference between the absolute values of the two correlator outputs.
- The error is low pass filtered and then applied to a voltage controlled Oscillator that controls the charging and discharging instants of the correlators.
- The instantaneous frequency of the local clock is advanced or retarded in an iterative manner under the equilibrium point is reached, and thereby symbol synchronization is achieved.

Frame synchronization

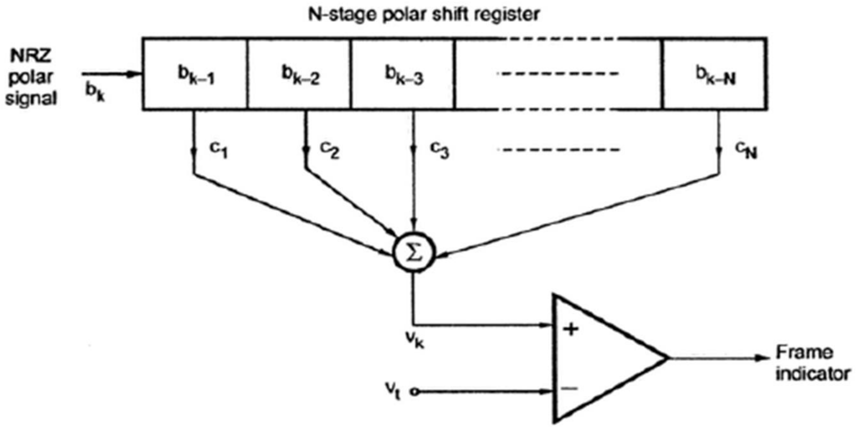
- In Time Division Multiplexing of data, the signal samples taken from each input channel forms a frame.
- The receiver has to know when a particular frame starts and when its individual message bit starts.
- This type of synchronization is called frame synchronization.
- For the frame synchronization sync bits are periodically inserted in the bit stream.



Time Division Multiplexing (TDM)



TDM frame sync format.



Block diagram of a frame synchronizer

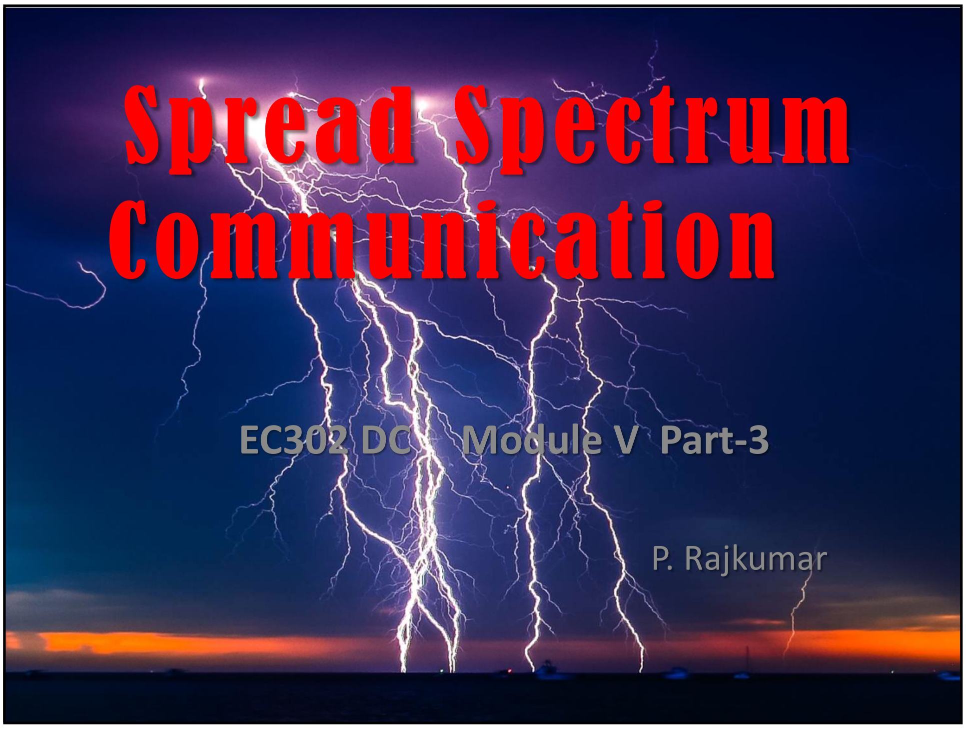
- The frame synchronizer uses polar shift registers.
- Received binary signal represented by NRZ polar signal.
- Thus b_k = -1 for 0 and b_k = +1 for 1.
- Let the sync words s_1, s_2, s_3,... s_N are to be detected.
- Let s_k be in polar forms +1 or -1.
- The tap gains are simply equal to the polar sync bits in reverse order.
- C_1=S_N, C_2=S_{N-2}, ...C_N=S_1.
- The tap gain outputs are added to give:

$$v_k = \sum_{i=1}^{N} c_i b_{k-i}$$

- If the received word is same as the sync word, then $c_i = b_{k-i}$
- $c_i b_{k-i}$=c_i^2=1 therefore

$$v_k = \sum_{i=1}^{N} 1 = N$$

- Threshold voltage of the comparator is set slightly below N.
- When the sync word is received, the comparator output goes high and it indicated the beginning of a new frame.



Spread Spectrum Communication

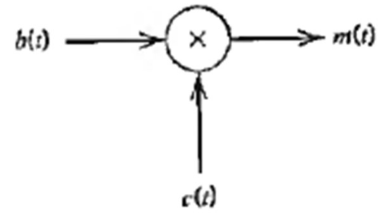
EC302 DC Module V Part-3

P. Rajkumar

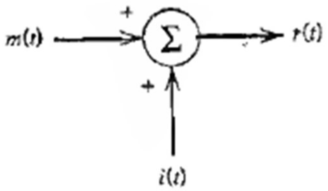
Spread-Spectrum Communication

- Spread-Spectrum Communication can provide protection against externally generated interfering (jamming) signals with finite power.
- The jamming signal may consist of a fairly powerful broadband noise or multitone waveform that is directed at the receiver for the purpose of disrupting communications.
- Protection against jamming waveforms is provided by purposely making the information-bearing signal occupy a bandwidth far in excess of the minimum bandwidth necessary to transmit it.
- This has the effect of making the transmitted signal assume a noiselike appearance so as to blend into the background.
- The transmitted signal is thus enabled to propagate through the channel undetected by anyone who may be listening.
- Therefore think of Spread Spectrum as a method of "camouflaging" the information-bearing signal.

- One method of widening the bandwidth of an information-bearing (data) sequence involves the use of modulation.
- Let $\{b_k\}$ denote a binary data sequence, and $\{c_k\}$ denote a pseudo-noise (PN) sequence.
- Let the waveforms b(t) and c(t) denote their respective polar Non Return-to-Zero representations in terms of two levels equal in amplitude and opposite in polarity, namely, ±1.
- Refer to b(t) is the information-bearing (data) signal, and c(t) is the PN signal.
- The desired modulation is achieved by applying the data signal b(t) and the PN signal c(t) to a product modulator or multiplier.
- Multiplication of two signals produces a signal whose spectrum equals the convolution of the spectra of the two component signals.
- Thus, if the message signal b(t) is narrowband and the PN signal c(t) is wideband, the product (modulated), signal m(t) will have a spectrum that is nearly the same as the wideband PN signal.
- Therefore, the PN sequence performs the role of a spreading code.



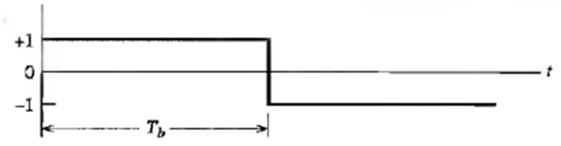
Idealized model of baseband spread-spectrum system Transmitter



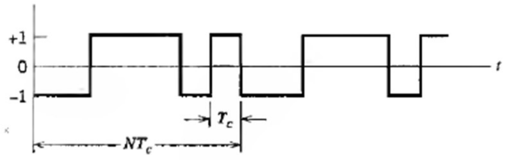
Channel

- By multiplying the information-bearing signal b(t) by the PN signal c(t), each information bit is “chopped” up into a number of small time increments.
- These small time increments are commonly referred to as Chips.
- For baseband transmission, the product signal m(t) represents the transmitted signal.
- Express the transmitted signal as,

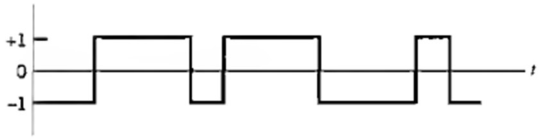
  m(t) = c(t).b(t) …….(8)



(a) Data signal $b(t)$



(b) Spreading code $c(t)$



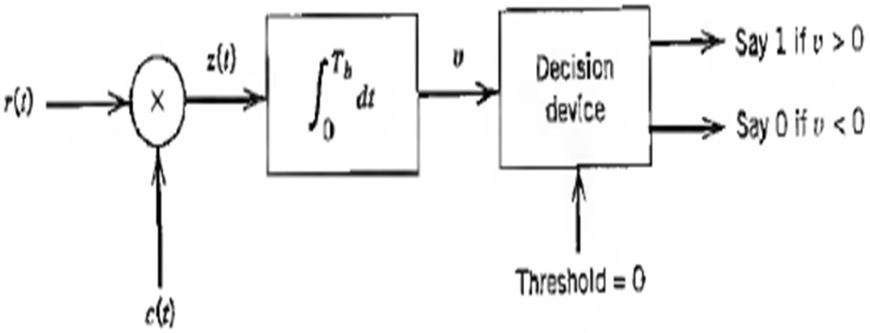
(c) Product signal $m(t)$

- The received signal r(t) consists of the transmitted signal m(t) plus an additive interference denoted by i(t), as shown in the channel model.

$$r(t) = m(t) + i(t) = c(t)b(t) + i(t) \quad ....(9)$$

- To recover the original message signal b(t), the received signal r(t) is applied to a demodulator that consists of a multiplier followed by an integrator, and a decision device.
- The multiplier is supplied with a locally generated PN sequence that is an exact replica of that used in the transmitter.
- The receiver operates in perfect synchronism with the transmitter, which means that the PN sequence in the receiver is lined up exactly with that in the transmitter.
- The multiplier output in the receiver is therefore given by,

$$z(t) = c(t)r(t) \quad = c^2(t)b(t) + c(t)i(t) \qquad ....(10)$$



Idealized model of baseband spread-spectrum system → Receiver.

- Eqn (10) shows that the data signal b(t) is multiplied twice by the PN signal c(t), whereas the unwanted signal i(t) is multiplied only once.
- The PN signal c(t) alternates between the levels -1 and +1, and the alternation is destroyed when it is squared.

$$c^2(t) = 1 \qquad \text{for all } t \quad \text{…….(10)}$$

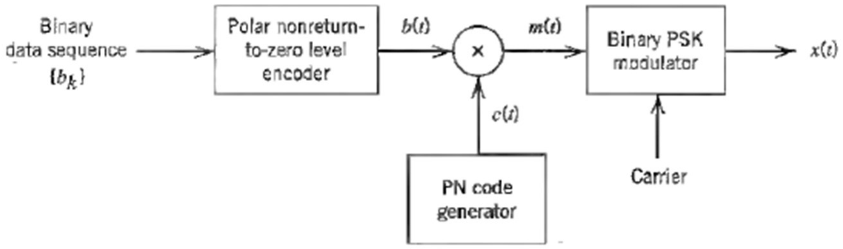
$$\therefore \; z(t) = b(t) + c(t)i(t) \quad \text{…….(11)}$$

- From eqn (11) that the data signal b(t) is reproduced at the multiplier output in the receiver, except for the effect of the interference represented by the additive term c(t).i(t).
- Multiplication of the interference i(t) by the locally generated PN signal c(t) means that the spreading code will affect the interference just as it did the original signal at the transmitter.
- The data component b(t) is narrowband, whereas the spurious component c(t).i(t) is wideband.
- Hence, by applying the multiplier output to a baseband (low-pass) filter with a bandwidth just large enough to accommodate the recovery of the data signal b(t), most of the power in the spurious component c(t).i(t) is filtered out.
- The effect of the interference i(t) is thus significantly reduced at the receiver.

- In the receiver shown, the low-pass filtering action is actually performed by the integrator that evaluates the area under the signal produced at the multiplier output.
- The integration is carried out for the bit interval $0 \leq t < T_b$, providing the sample value v.
- Finally, a decision is made by the receiver:
- If v is greater than the threshold of zero the receiver says that binary symbol 1 of the original data sequence was sent in the interval $0 \leq t < T_b$
- If v is less than zero, the receiver says that symbol 0 was sent.
- If v is exactly zero the receiver makes a random guess in favor of 1 or 0.

- The use of a Spreading Code (with Pseudo-Random Noise-PN properties) in the Transmitter produces a wideband transmitted signal that appears noiselike to a Receiver that has no knowledge of the spreading code.
- For a prescribed data rate, the longer we make the period of the spreading code, the closer will the transmitted signal be to a truly random binary wave, and the harder it is to detect.
- The price to pay for the improved protection against interference is <u>increased transmission bandwidth</u>, <u>system complexity</u>, and <u>processing delay</u>. NO PAIN NO GAIN!
- However, when the primary concern is the <u>Security</u> of transmission, these are not unreasonable costs to pay.

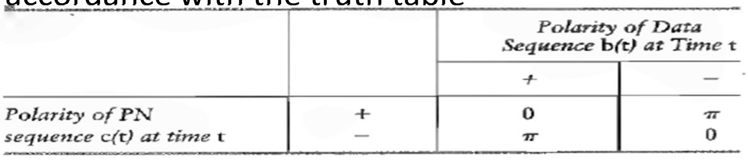
Direct-Sequence Spread Spectrum with Coherent Binary Phase-Shift Keying



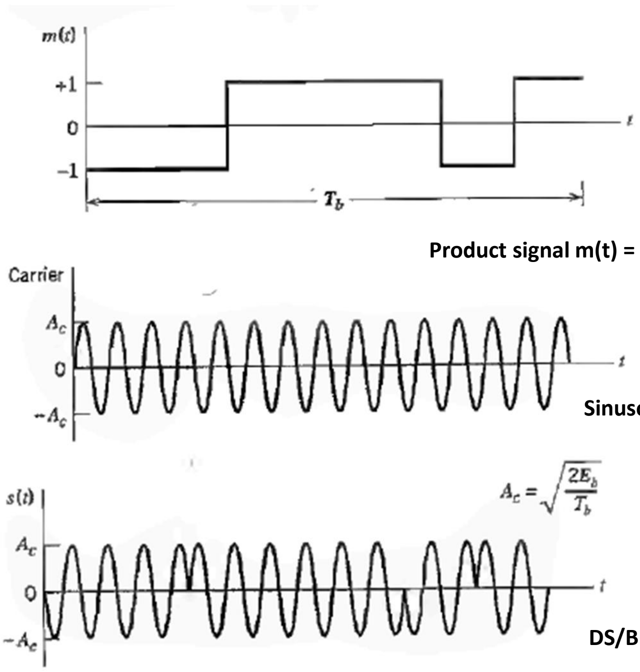
Direct-sequence spread coherent phase-shift keying Transmitter

- To provide for the use of **Direct-Sequence Spread Spectrum** in passband transmission over a satellite channel, for example, incorporate coherent binary phase-shift keying (PSK) into the transmission.

- The transmitter first converts the incoming binary data sequence *{b_k}* into a polar NRZ waveform *b(t)*, which is followed by two stages of modulation.
- The <u>first stage</u> consists of a <u>product modulator</u> or multiplier with the data signal b(t) (representing a data sequence) and PN signal c(t) (representing the PN sequence) as inputs.
- The second stage consists of a binary PSK modulator.
- The transmitted signal x(t) is thus a Direct-Sequence Spread Binary Phase-Shift-Keyed (DS/BPSK) signal.
- The phase π, depending on the polarities of the message signal *b(t)* and PN signal *c(t)* at time *t* in accordance with the truth table



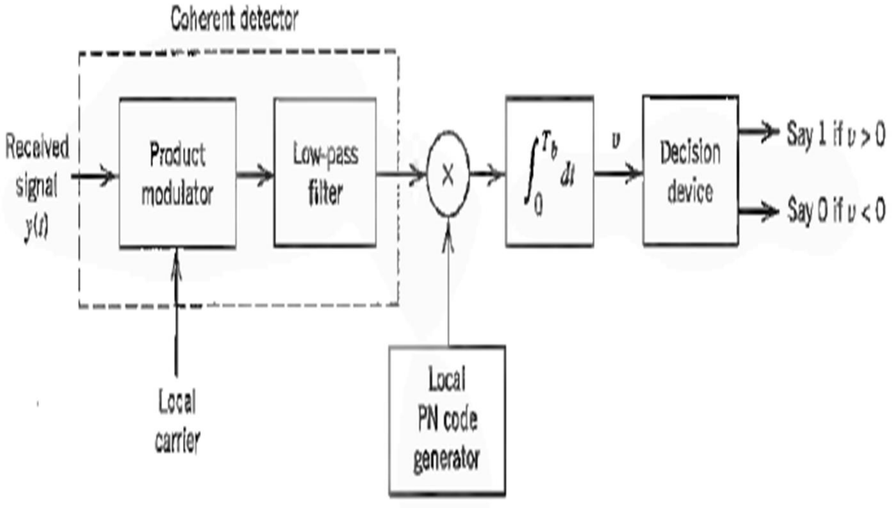
		Polarity of Data Sequence b(t) *at Time* t	
		+	—
Polarity of PN sequence c(t) *at time* t	+	0	π
	—	π	0



Product signal m(t) = c(t).b(t)

Sinusoidal carrier

DS/BPSK signal



Direct-sequence spread coherent phase-shift keying Receiver

Direct-sequence spread coherent phase-shift keying Receiver

- The receiver consists of two stages of demodulation.
- In the first stage, the received signal y(t) and a locally generated carrier are applied to a product modulator followed by a low-pass filter whose bandwidth is equal to that of the original message signal m(t).
- This stage of the demodulation process reverses the phase-shift keying applied to the transmitted signal.
- The second stage of demodulation performs spectrum despreading by multiplying the low-pass filter output by a locally generated replica of the PN signal c(t), followed by integration over a bit interval $0 \leq t \leq T_b$ and finally decision making.

Signal-Space Dimensionality and Processing Gain

- Develop the signal space representation of the Transmitted signal and Interfering signal (Jammer).
- Consider the set of orthonormal basis functions:

$$\phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}} \cos(2\pi f_c t), & kT_c \leq t \leq (k+1)T_c \\ 0, & \text{otherwise} \end{cases} \quad \text{.......(12)}$$

$$\tilde{\phi}_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}} \sin(2\pi f_c t), & kT_c \leq t \leq (k+1)T_c \\ 0, & \text{otherwise} \end{cases} \quad \text{.......(13)}$$

$$k = 0, 1, \ldots, N-1$$

- T_c is the chip duration, N →number of chips per bit.

- The transmitted signal x(t) for the interval of an information bit is:

$$\begin{aligned} x(t) &= c(t)s(t) \\ &= \pm\sqrt{\frac{2E_b}{T_b}}\, c(t)\cos(2\pi f_c t) \qquad \text{.......(14)} \\ &= \pm\sqrt{\frac{E_b}{N}} \sum_{k=0}^{N-1} c_k \phi_k(t), \qquad 0 \le t \le T_b \end{aligned}$$

- Where E_b is the signal energy per bit.
- The plus sign correspond s to information bit 1 , and the minus sign correspond s to information bit 0 .

- The code sequence $\{c_0, c_1, \ldots, c_{N-1}\}$ denotes the PN sequence, with $c_k = \pm 1$.
- The transmitted signal x(t) is therefore N dimensional in that it requires a minimum of N orthonormal functions for its representation.

Consider the representation of the interfering signal j(t).

- The jammer can hope to know the transmitted signal bandwidth.
- But there is no way that the jammer can have the knowledge of the signal phase.

- In general form the jammer can be represented by

$$j(t) = \sum_{k=0}^{N-1} j_k\phi_k(t) + \sum_{k=0}^{N-1} \tilde{j}_k\tilde{\phi}_k(t), \qquad 0 \le t \le T_b$$

…….(15)

where

$$j_k = \int_0^{T_b} j(t)\phi_k(t)\,dt, \qquad k = 0, 1, \ldots, N-1$$

…….(16)

and

$$\tilde{j}_k = \int_0^{T_b} j(t)\tilde{\phi}_k(t)\,dt, \qquad k = 0, 1, \ldots, N-1$$

…….(17)

- Thus the interference j(t) is 2N dimensional; that is it has twice the number of dimensions required for representing the transmitted DS/BPSK signal x(t).
- The average power of the interference signal j(t)

$$\begin{aligned} J &= \frac{1}{T_b}\int_0^{T_b} j^2(t)\,dt \\ &= \frac{1}{T_b}\sum_{k=0}^{N-1} j_k^2 + \frac{1}{T_b}\sum_{k=0}^{N-1} \tilde{j}_k^2 \end{aligned}$$

…….(18)

- Due to the lack of knowledge of signal phase, the best strategy a jammer can apply is to place equal energy in the cosine and sine co ordinates.

- Hence we can assume

$$\sum_{k=0}^{N-1} j_k^2 = \sum_{k=0}^{N-1} \tilde{j}_k^2 \qquad ....(19)$$

- Now $$J = \frac{2}{T_b} \sum_{k=0}^{N-1} j_k^2 \qquad ....(20)$$

- The coherent detector output is

$$v = \sqrt{\frac{2}{T_b}} \int_0^{T_b} u(t) \cos(2\pi f_c t)\, dt$$
$$= v_s + v_{cj} \qquad ....(21)$$

- Where the components v_s and v_{cj} are due to the despread binary PSK signal s(t) and the spread interference c(t)j(t) respectively.

- These two components are defined as

$$v_s = \sqrt{\frac{2}{T_b}} \int_0^{T_b} s(t) \cos(2\pi f_c t)\, dt \qquad ....(22)$$

$$v_{cj} = \sqrt{\frac{2}{T_b}} \int_0^{T_b} c(t) j(t) \cos(2\pi f_c t)\, dt \qquad ....(23)$$

- The despread binary PSK signal s(t) is

$$s(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), \qquad 0 \leq t \leq T_b \qquad ....(24)$$

- Where the plus sign corresponds to information bit 1 and minus corresponds to 0.

- Assume the carrier frequency f_c is the integer multiple of 1/Tb. Then we have

$$v_s = \pm\sqrt{E_b}$$ …….(25)

- Consider the next component v_{cj}

$$v_{cj} = \sqrt{\frac{2}{T_b}} \sum_{k=0}^{N-1} c_k \int_{kT_c}^{(k+1)T_c} j(t) \cos(2\pi f_c t)\, dt$$ …….(26)

- Sub

$$\phi_k(t) = \begin{cases} \sqrt{\frac{2}{T_c}} \cos(2\pi f_c t), & kT_c \leq t \leq (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$ (7.14)(27)

$$j_k = \int_0^{T_b} j(t)\phi_k(t)\, dt, \quad k = 0, 1, \ldots, N-1$$ …….(28)

- Substituting the equation for ϕ_k(t) and j_k, we can redefine v_{cj}

$$\begin{aligned} v_{cj} &= \sqrt{\frac{T_c}{T_b}} \sum_{k=0}^{N-1} c_k \int_0^{T_b} j(t)\phi_k(t)\, dt \\ &= \sqrt{\frac{T_c}{T_b}} \sum_{k=0}^{N-1} c_k j_k \end{aligned}$$ …….(29)

- Where N =$\frac{T_b}{T_c}$
- We now approximate the PN sequence as independent and identically distributed sequence

$$V_{cj} = \sqrt{\frac{T_c}{T_b}} \sum_{k=0}^{N-1} C_k j_k$$ …….(30)

- In the above equation the jammer is assumed to be fixed.
- Where C_k is a random variable with sample values c_k
- Thus V_{cj} is a random variable with sample values v_{cj}
- With the C_k treated as i.i.d random variable.
- The probability of the event C_k = ±1 is

$$P(C_k = 1) = P(C_k = -1) = \tfrac{1}{2} \qquad \text{.......(31)}$$

- Accordingly the mean of the random variable V_{cj} is zero for fixed k we have

$$\begin{aligned} E[C_k j_k | j_k] &= j_k P(C_k = 1) - j_k P(C_k = -1) \\ &= \tfrac{1}{2} j_k - \tfrac{1}{2} j_k \\ &= 0 \end{aligned} \qquad \text{.......(32)}$$

- For a fixed vector j, representing the set of coefficients, j_0, j_1,..... j_{N-1} the variance of V_c is given by
- $\text{Var}(x)=\sum \frac{(x-\mu)^2}{N}$

$$\text{var}[V_{cj} | \mathbf{j}] = \frac{1}{N} \sum_{k=0}^{N-1} j_k^2 \qquad \text{.......(33)}$$

- Since the spread factor N = T_b/T_c

- Variance can be expressed in terms of the average interference (jammer) power J, where

$$J = \frac{2}{T_b} \sum_{k=0}^{N-1} j_k^2$$

$$\text{var}[V_{cj}|\mathbf{j}] = \frac{JT_c}{2} \qquad \text{.......(34)}$$

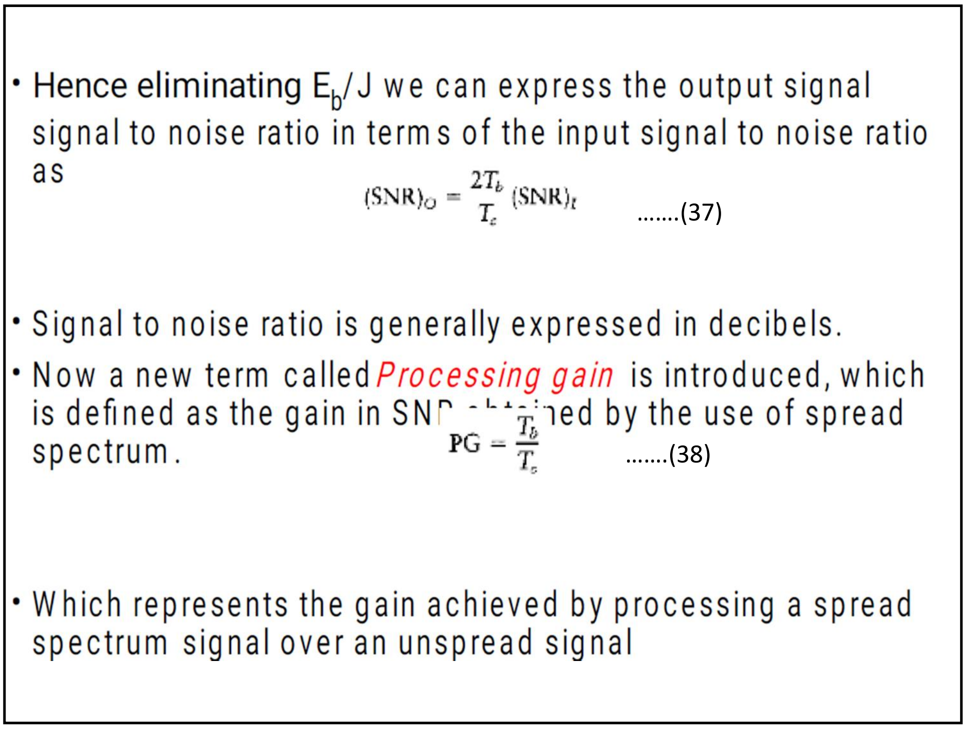
- Thus the random V_{cj} variable has zero mean and variance $JT_c/2$
- The signal component at the coherent detector output (during each bit interval) equals ± $\sqrt{E_b}$, where E_b is the signal energy per bit.
- Hence the peak instantaneous power of the signal component is E_b.

- The output signal to noise ratio can be defined as the instantaneous peak power E_b divided by the variance of the equivalent noise component

$$(\text{SNR})_O = \frac{2E_b}{JT_c} \qquad \text{.......(35)}$$

- The average signal power at the receiver input equals E_b/T_b.
- Thus input signal to noise ratio can be defined as

$$(\text{SNR})_I = \frac{E_b/T_b}{J} \qquad \text{.......(36)}$$



- Hence eliminating E_b/J we can express the output signal signal to noise ratio in terms of the input signal to noise ratio as

$$(\mathrm{SNR})_O = \frac{2T_b}{T_c}(\mathrm{SNR})_I \qquad ....(37)$$

- Signal to noise ratio is generally expressed in decibels.
- Now a new term called *Processing gain* is introduced, which is defined as the gain in SNR obtained by the use of spread spectrum.

$$\mathrm{PG} = \frac{T_b}{T_c} \qquad ....(38)$$

- Which represents the gain achieved by processing a spread spectrum signal over an unspread signal

Digital Communication over Fading Multipath Channels

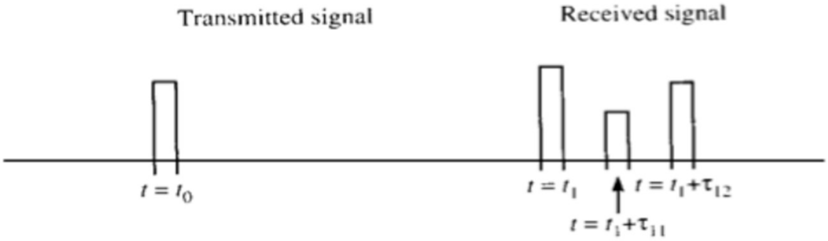
EC 302 Digital Communication Module VI

Introduction

- Channels with random time variant impulse responses.
- Model for shortwave ionospheric radio channels 3- 30 MHz (HF), tropospheric scatter, ionospheric forward scatter 30-300 MHz(VHF), (BTH) radio channels 300-3000MHz (UHF), 3 – 30 GHz (SHF)
- Time variant impulse responses of these channels are due to constantly changing physical characteristics of the media.
- Try to improve the badly impaired SNR due to fading by efficient mod/coding and demod/decoding techniques.

Characterization of Fading Multipath Channels

- Time spread introduced on the signal *s(t)* transmitted through multipath channel.



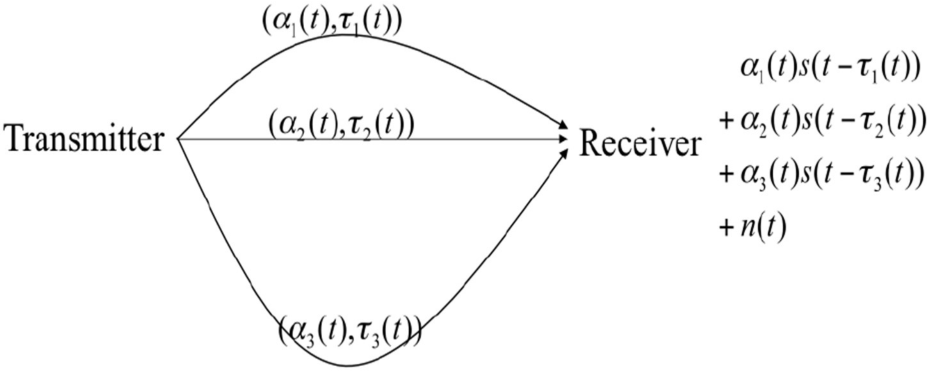
- Time variations in the structure of the medium.
  - Nature of multipath varies with time

Time spread phenomenon of multipath channels

- (Unpredictable) Time-variant factors
- Delay
- Number of spreads
- Size of the receive pulses

Characterization of Fading Multipath Channels

- The multipath fading channels with additive noise
- Each path has a tv prop delay and tv atten factor.



$$\alpha_1(t)s(t-\tau_1(t)) + \alpha_2(t)s(t-\tau_2(t)) + \alpha_3(t)s(t-\tau_3(t)) + n(t)$$

- Transmitted signal

$$s(t) = \mathbf{Re}\left\{ s_\ell(t) e^{\imath 2\pi f_c t} \right\} \quad 1$$

- Received band pass signal in absence of additive noise

$$x(t) = \sum_n \alpha_n(t) s[t - \tau_n(t)] \quad 2$$

- where $\alpha_n(t)$ is the attenuation factor for the signal received on the nth path and $\tau_n(t)$ is the propagation delay for the nth path.

$$x(t) = \mathrm{Re}\left(\left\{ \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} s_l[t - \tau_n(t)] \right\} e^{j2\pi f_c t} \right)$$

3

- In the absence of noise the equivalent low pass received signal is

$$r_l(t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} s_l[t - \tau_n(t)] \tag{4}$$

- $r_l(t)$ is the response of an equivalent low pass channel to the equivalent low pass signal $s_l(t)$.
- It follows that the equivalent low pass channel is described by the time-variant impulse response

$$c(\tau; t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} \delta[\tau - \tau_n(t)] \tag{5}$$

- For tropospheric scatter channel, the received signal consists of a continuous multipath components.

$$x(t) = \int_{-\infty}^{\infty} \alpha(\tau; t) s(t - \tau) d\tau \tag{6}$$

- where $\alpha(\tau ; t)$ denotes the attenuation of the signal components at delay τ and at time t

$$x(t) = \text{Re}\left\{\left[\int_{-\infty}^{\infty} \alpha(\tau; t) e^{-j2\pi f_c \tau} s_l(t - \tau)\, d\tau\right] e^{j2\pi f_c t}\right\} \tag{7}$$

- represents the convolution of $s_l(t)$ with an equivalent low pass time-variant impulse response $c(\tau ; t)$, it follows that

$$c(\tau; t) = \alpha(\tau; t) e^{-j2\pi f_c \tau} \tag{8}$$

- where *c*(*τ* ; *t*) represents the response of the channel at time *t* due to an impulse applied at time *t* −*τ.*
- defn of the equiv low pass impulse response when channel results in continuous multipath.
- Time-variant impulse response *c*(*τ* ; *t*) is a complex-valued Gaussian r p in the *t* variable.
- The ampltd variations in the rx signal, termed *signal fading*, are due to the time-variant multipath characteristics of the channel.

- If *c*(*τ* ; *t*) is modeled as a zero-mean complex Gaussian process, the envelope |*c*(*τ* ; *t*)| at any instant *t* is Rayleigh-distributed and the channel is said to be a Rayleigh fading channel.
- If *c*(*τ* ; *t*) can no longer be modeled as having zero-mean, the envelope |*c*(*τ* ; *t*)| has a Rice distribution and the channel is said to be a Ricean fading channel.
- Another probability distribution function that has been used to model the envelope of fading signals is the Nakagami-*m* distribution.

Channel Correlation Functions and Power Spectra

- The equiv low pass impulse response $c(\tau\,;t)$, is complex-valued random process in the *t* variable.
- Assume $c(\tau\,;t)$ is wide sense stationary (WSS).
- Then the Autocorrelation function of $c(\tau\,;t)$ is:

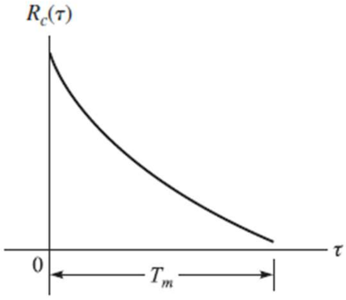
$$R_c(\tau_2, \tau_1; \Delta t) = E\left[c^*(\tau_1; t)c(\tau_2; t + \Delta t)\right] \quad \mathbf{9}$$

- Atten and phase shift of channel asso with path delay τ_1 is uncorr with the atten and phase shift asso with path delay τ_2. -> Uncorrelated Scattering (**US**).

- Use the assumption that the scattering at two different delays is uncorrelated:

$$E\left[c^*(\tau_1; t)c(\tau_2; t + \Delta t)\right] = R_c(\tau_1; \Delta t)\delta(\tau_2 - \tau_1) \quad \mathbf{10}$$

- If $\Delta t = 0$, resulting $R_c(\tau\,; 0) \equiv R_c(\tau\,)$ is simply the av power o/p of channel as a fn of time delay τ .
- So $R_c(\tau\,)$ is called the Multipath Intensity profile or the Delay Power spectrum of the channel.
- In general, $R_c(\tau\,; \Delta t)$ gives the av power o/p as a fn of time delay τ and the diff Δt in obsrvn time.



- The range of values of τ over which $R_c(\tau)$ is essentially nonzero is called the *multipath spread of the channel* and is denoted by T_m.

Frequency domain characterization

- Fourier transform of c(τ ; t),gives the tv transfer function C(f ; t),

$$C(f;t) = \int_{-\infty}^{\infty} c(\tau;t)e^{-j2\pi f\tau}\,d\tau \qquad \textbf{11}$$

- Assuming **US- WSS** channel, the autocorrelation function:

$$R_C(f_2, f_1; \Delta t) = E\left[C^*(f_1;t)C(f_2;t+\Delta t)\right] \qquad \textbf{12}$$

- $R_C(f_2, f_1; \Delta t)$ is related to $R_c(\tau ; \Delta t)$ by the Fourier transform as shown below:

$$R_C(f_2, f_1; \Delta t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[c^*(\tau_1; t)c(\tau_2; t+\Delta t)\right] e^{j2\pi(f_1\tau_1 - f_2\tau_2)} d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_c(\tau_1; \Delta t)\delta(\tau_2 - \tau_1)e^{j2\pi(f_1\tau_1 - f_2\tau_2)} d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} R_c(\tau_1; \Delta t)e^{j2\pi(f_1 - f_2)\tau_1} d\tau_1$$

$$= \int_{-\infty}^{\infty} R_c(\tau_1; \Delta t)e^{-j2\pi\Delta f\,\tau_1} d\tau_1 \equiv R_C(\Delta f; \Delta t)$$

- where $\Delta f = f_2 - f_1$.

- $R_C(\Delta f ; \Delta t)$ is the Fourier transform of the Multipath Intensity profile.
- US implies that the autocorr fn of $C(f ; t)$ in frequency is a function of only the frequency difference $\Delta f = f_2 - f_1$.
- $R_C(\Delta f ; \Delta t)$ →the <u>spaced-frequency, spaced time correlation function of the channel</u>.
- $t = 0$ →$R_C(\Delta f ; 0) \equiv R_C(\Delta f)$
- $R_c(\tau ; 0) \equiv R_c(\tau)$, the transform relationship is simply

$$R_C(\Delta f) = \int_{-\infty}^{\infty} R_c(\tau)e^{-j2\pi\Delta f\,\tau} d\tau$$

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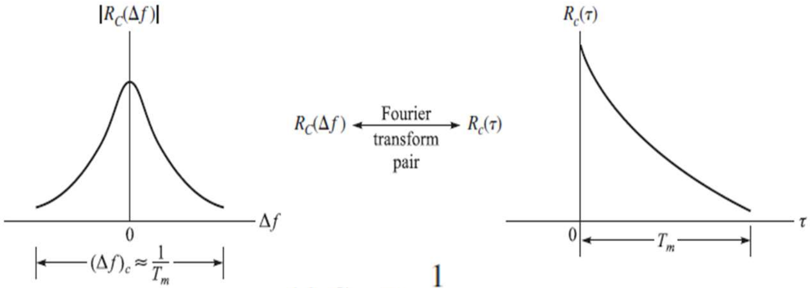
Relationship between $R_C(\Delta f)$ and $R_c(\tau)$



Spaced-frequency correlation function

$$(\Delta f)_c \approx \frac{1}{T_m}$$

14

Multipath intensity profile

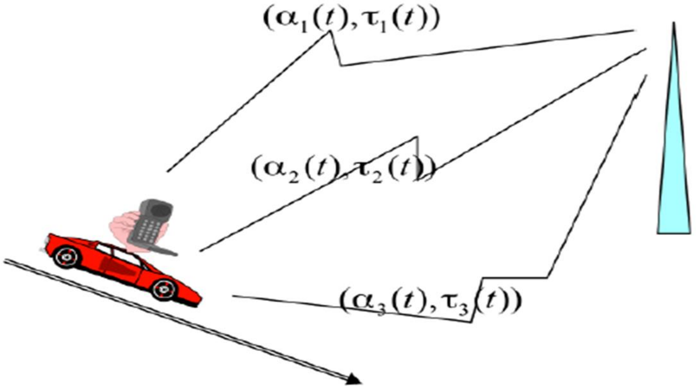
- the reciprocal of the multipath spread is a measure of the <u>coherence bandwidth</u> of the channel.

Implications

- Two sinusoids with freq separation greater than coherence BW are affected differently by channel.
- If the coherence BW is small compared to BW of tx signal, the channel is <u>frequency selective</u>.
- The signal is severely distorted by the channel.
- If coherence BW is large compared to BW of tx signal, the channel is <u>frequency nonselective</u>.

Time varying characterization: Doppler effect

- Doppler effect appears via the argument Δt.



Doppler Power Spectrum of a US-WSS channel

- To relate Doppler effects to time variations of channel, Fourier transform of $R_C(\Delta f;\Delta t)$ w.r.t variable Δt is the function $S_C(\Delta f\,;\lambda)$.
- The Doppler power spectrum is:

$$\mathcal{S}_C(\Delta f;\lambda)=\int_{-\infty}^{\infty}R_C(\Delta f;\,\Delta t)e^{-j2\pi\lambda\,\Delta t}d\,\Delta t \qquad \textbf{15}$$

- With Δf set to zero → $S_C(0;\lambda)\equiv S_C(\lambda)$,

$$\mathcal{S}_C(\lambda)=\int_{-\infty}^{\infty}R_C(0;\,\Delta t)e^{-j2\pi\lambda\,\Delta t}d\,\Delta t \qquad \textbf{16}$$

- The function $S_C(\Delta f\,;\lambda)$. is a power spectrum that gives signal intensity as a function of Doppler frequency λ.
- This is the Doppler Power Spectrum of the channel.

Doppler Spread of the channel

- The values of λ over which $S_C(\lambda)$ is nonzero is called Doppler spread B_d of channel.
- Since $S_C(\lambda)$ is related to $R_C(\Delta t)$ by Fourier transform, reciprocal of B_d is a measure of coherence time of channel. That is,

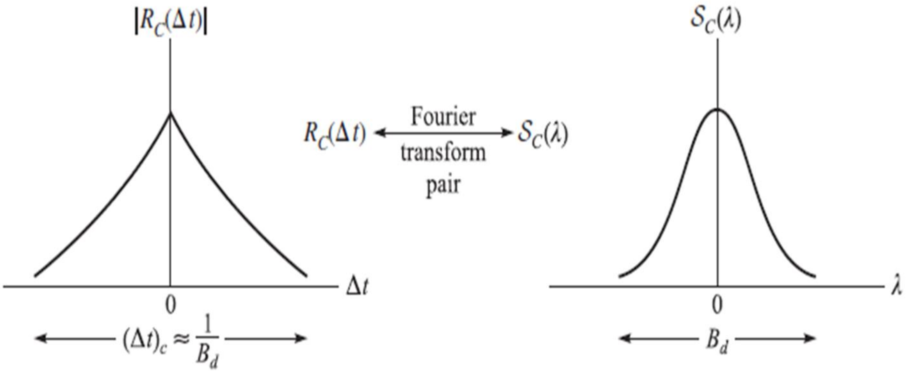
$$(\Delta t)_c \approx \frac{1}{B_d} \quad 17$$

- Slowly changing channel has a large coherence time which implies a small Doppler spread.

Types of Fading

- If symbol period $T > (\Delta t)_c$, the channel is classified as Fast Fading.
- i.e., channel statistics changes within one symbol!
- If symbol period $T < (\Delta t)_c$, the channel is classified as Slow Fading.

Relationship between $R_C(\Delta t)$ and $S_C(\lambda)$



Spaced-time correlation fnction

Doppler power spectrum

Scattering function of the channel

- Define a new function, denoted by S(τ ; λ), to be the Fourier transform of $R_c(\tau\,;\,\Delta t)$ in Δt var,

$$S(\tau;\lambda) = \int_{-\infty}^{\infty} R_c(\tau;\,\Delta t)e^{-j2\pi\lambda\,\Delta t}d\Delta t \qquad \textbf{18}$$

- It follows that $S(\tau\,;\,\lambda)$ and $S_C(f\,;\,\lambda)$ are a Fourier transform pair. i.e.,

$$S(\tau;\lambda) = \int_{-\infty}^{\infty} S_C(\Delta f;\lambda)e^{\,j2\pi\tau\,\Delta f}d\Delta f \qquad \textbf{19}$$

- Furthermore, $S(\tau\,;\lambda)$ and $R_C(\Delta f\,;\Delta t)$ are related by the double Fourier transform,

$$S(\tau;\lambda)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_C(\Delta f;\Delta t)e^{-j2\pi\lambda\,\Delta t}e^{\,j2\pi\tau\,\Delta f}\,d\Delta t\,d\Delta f$$

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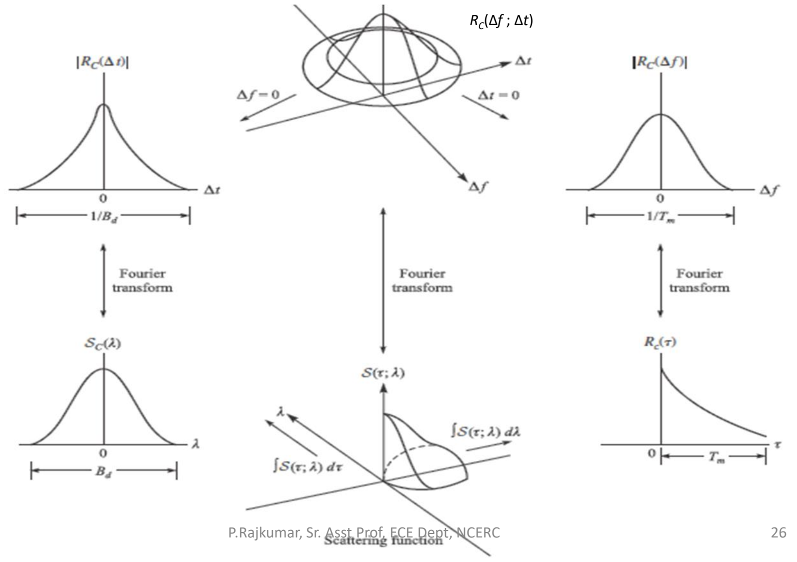
- Scattering function provides a measure of average power output of channel as a function of time delay *τ* and Doppler frequency *λ*.

Relationships between $R_C(\Delta f\,;\Delta t)$, $R_c(\tau\,;t)$, $S_C(\Delta f\,;\lambda)$, and $S(\tau\,;\lambda)$



Scattering function

Statistical Models for Fading Channels

- Different prob distribs are used model fading channels.
- If many Scatterers in channel contribute to signal at rx, as is case of ionospheric or tropospheric signal propagation, central limit theorem leads to a Gaussian process model for channel impulse response.
- If process is zero-mean, then envelope of channel response at any time inst has a Rayleigh probability distribution and phase is uniformly distrib in interval (0, 2π).
- Another model for envelope of channel response is 2 parameter Nakagami-m distribution.
- The Rice distribution is also a 2 parameter distribution.
- Fading occurs in LOS commn links with multipath components arising from sec reflections, or signal paths, from surrounding terrain, no: of multipath components is small→ channel may be modeled in a somewhat simpler form.

- Consider airplane to grnd commn link in which there is direct path and a single multipath component at a delay t_0 relative to direct path.
- Impulse response of such a channel may be modeled as:

$$c(\tau;t) = \alpha\delta(\tau) + \beta(t)\delta[\tau - \tau_0(t)] \qquad \textbf{21a}$$

- α is the attenuation factor of the direct path and β(t) represents the tv multipath signal component resulting from terrain reflections, can be characterized as a zero-mean Gaussian rp. The TF for this channel model is:

$$C(f;t) = \alpha + \beta(t)e^{-j2\pi f\tau_0(t)} \qquad \textbf{21b}$$

- This channel fits Ricean fading model.
- The direct path has attenuation α and $\beta(t)$ represents the Rayleigh fading component.

Rummler's model

- The diff delay on two multipath components is relatively small, model developed by Rummler has a channel TF

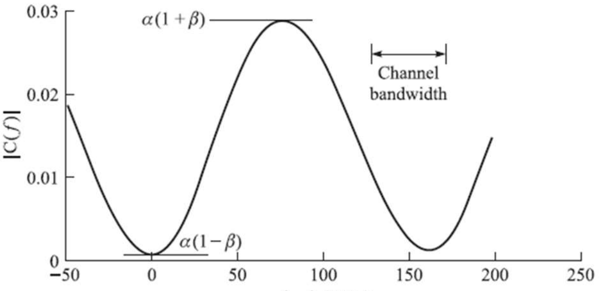
$$C(f) = \alpha[1 - \beta e^{-j2\pi(f-f_0)\tau_0}] \qquad \mathbf{22}$$

- α is overall atten parameter, β is called a shape parameter due to the multipath components, f_0 is freq of fade min, and τ_0 is relative time delay between direct and multipath components.
- The magnitude-square response of C(f) is

$$|C(f)|^2 = \alpha^2[1 + \beta^2 - 2\beta\cos 2\pi(f - f_0)\tau_0] \qquad \mathbf{23}$$

Plot of magnitude-square response of C(f)

- τ_0 = 6.3 ns.
- effect of multipath component is to create a deep attenuation at $f = f_0$ and at multiples of $1/\tau_0 \approx 159$ MHz. By comparison, the typical channel bw is 30 MHz.



Propagation models for mobile radio channels

- Path loss of radio waves prop thro’ free space is inv prop to d^2, where d is distance between tx and rx.
- But in mobile radio channel propagation is generally neither free space nor line of sight.
- The mean path loss in mobile radio channels may be char as being inv prop to d^{p}, where $2 \le p \le 4$, with d^{4} being a worst-case model.
- So the path loss is usually much more severe compared to that of free space.

- There are a no:of factors affecting path loss in mobile radio comm.
- Among these factors are *base station antenna height*, *mobile antenna height*, *operating frequency*, *atmospheric conditions*, and *presence or absence of buildings and trees*.
- Various mean path loss models have been developed that incorporate such factors.
- For ex, a model for large city in urban area is the Hata model, in which the mean path loss is expressed as

$$\text{Loss in dB} = 69.55 + 26.16\log_{10} f - 13.82\log_{10} h_t - a(h_r) + (44.9 - 6.55\log_{10} h_t)\log_{10} d$$

24

- where f is op freq in MHz (150 < f < 1500), h_t is tx antenna height in meters (30 < h_t < 200), h_r is rx antenna height in meters (1 < h_r < 10), d is distance between tx and rx in km (1 < d < 20),

$$a(h_r) = 3.2(\log_{10} 11.75h_r)^2 - 4.97, \qquad f \geq 400\,\text{MHz} \quad 25$$

- Another problem with mobile radio prop is effect of shadowing of signal due to large obstructions, such as large buildings, trees, and hilly terrain between the tx and the rx.
- Shadowing is usually modeled as a multiplicative and slowly time varying rp. i.e., the rx signal may be characterized as

$$r(t) = A_0 g(t) s(t) \quad 26$$

- A_0 represents mean path loss, s(t) is tx signal, and g(t) is a rp that represents shadowing effect.
- At any time instant, the shadowing process is modeled statistically as lognormally distrib. The pdf for the lognormal distribution is

$$p(g) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2}\, g} e^{-(\ln g - \mu)^2/2\sigma^2} & (g \geq 0) \\ 0 & (g < 0) \end{cases} \quad 27$$

- Use a new rv X such that X = ln g, then

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty \quad 28$$

- rv X represents path loss in dB, μ is mean path loss in dB, and σ is std dev of path loss in dB.
- For typical cellular and microcellular environs, σ is in range of 5–12 dB.

Model Based Signal Characteristics

- Let $s_l(t)$ be the equi lp signal tx over channel and let $S_l(f)$ denote its freq content.
- Then the equi lp rx signal, exclusive of additive noise, may be expr in terms of c(τ ; t) and s_l (t)

$$r_l(t) = \int_{-\infty}^{\infty} c(\tau; t) s_l(t - \tau)\, d\tau \qquad \textbf{29}$$

- or in terms of the frequency functions C(f ; t) and S_l (f)

$$r_l(t) = \int_{-\infty}^{\infty} C(f; t) S_l(f) e^{j2\pi f t}\, df \qquad \textbf{30}$$

- The tv channel char by TF C(f ; t) distorts signal S_l(f).
- If S_l(f) has a BW, W > coherence BW $(\Delta f)_c$ of channel, S_l(f) is subjected to different gains and phase shifts across the band.
- In such a case, channel is said to be <u>Frequency-selective</u>.
- Additional distortion is caused by the tv_s in C(f ; t) is a variation in rx signal strength, and has been termed <u>Fading</u>.
- Frequency selectivity and Fading are two different types of signal distortion.
- Frequency selectivity depends on <u>multipath spread</u> or, <u>coherence bandwidth</u> of the channel relative to the tx signal BW.
- Fading depends on the tv_s of channel, which are characterized by the <u>coherence time</u> $(\Delta t)_c$ or, the <u>Doppler spread</u> Bd .

- Effect of channel on tx signal $s_l(t)$ is a fn of signal bw and signal duration.
- If signaling interval T to satisfy the condition $T >> T_m$, channel introduces a negligible intersymbol interference.
- If bw of signal pulse $s_l(t)$ is $W \approx 1/T$, $T >> Tm$ →

$$W \ll \frac{1}{T_m} \approx (\Delta f)_c \qquad \mathbf{31}$$

- The channel is *frequency-nonselective*.
- So all freq components in $S_l(f)$ undergo same atten and phase shift in tx thro' channel.
- But this implies that, within the bw occupied by $S_l(f)$, tv TF $C(f\,;t)$ of channel is a complex-valued constant in f.

- Since $S_l(f)$ has its freq content concentrated near $f = 0$, $C(f\,;t) = C(0;t)$.→
- Eqn 30 reduces to

$$r_l(t) = C(0;t)\int_{-\infty}^{\infty} S_l(f)e^{j2\pi ft}\,df$$
$$= C(0;t)s_l(t) \qquad \mathbf{32}$$

- So when signal bw W is << $(\Delta f)_c$ of channel, rx signal is tx signal mult by a complex-valued rp $C(0;t)$, which represents tv char of channel.
- The TF C(0; t) for a freq -nonselective channel is:

$$C(0;t) = \alpha(t)e^{j\phi(t)} \qquad \mathbf{33}$$

- $\alpha(t)$ is envelope and $\phi(t)$ is phase of equi lp channel.
- When $C(0; t)$ is modeled as a zero-mean complex-valued Gaussian rp, envelope $\alpha(t)$ is Rayleigh-distributed for any fixed value of t, $\phi(t)$ is uniformly distr over interval $(-\pi, \pi)$.
- Rapidity of fading on freq-nonselective channel is determined either from Correlation fn $R_C(t)$ or from Doppler power spectrum $S_C(\lambda)$.
- Either of channel parameters $(\Delta t)_c$ or B_d can be used to characterize the rapidity of the fading.

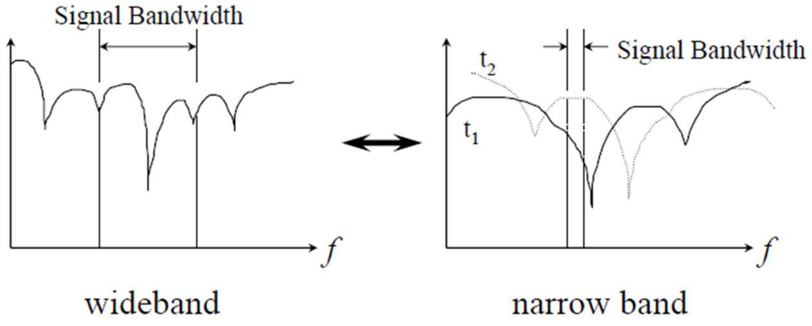
- Suppose it is possible to select signal bw W to satisfy condition $W << (\Delta f)_c$ and the signaling interval T to satisfy condition $T<<(\Delta t)_c$.
- Since T << coherence time of channel, channel atten and phase shift are fixed for duration of at least one signaling interval. When this condition holds, call the channel a slowly fading channel.
- When $W \approx 1/T$, channel be frequency-nonselective and slowly fading →product of T_m and B_d must satisfy condition $T_m . B_d < 1$.

- The product $T_m B_d$ is called the Spread factor of the channel.
- If $T_m B_d < 1$, channel is said to be Under-spread;
- otherwise, it is Over-spread

Types of Small-Scale Fading

- Based on multi-path time delay spread
  - Flat Fading (narrowband system)
    - BW of signal < BW of channel
    - Delay spread < Symbol period
  - Frequency Selective Fading (wideband system)
    - BW of signal > BW of channel
    - Delay spread > Symbol period

Wideband v.s. Narrowband



Flat Fading

- Signal undergoes flat fading if W is << $(\Delta f)_c$ & $T_s < T_m$

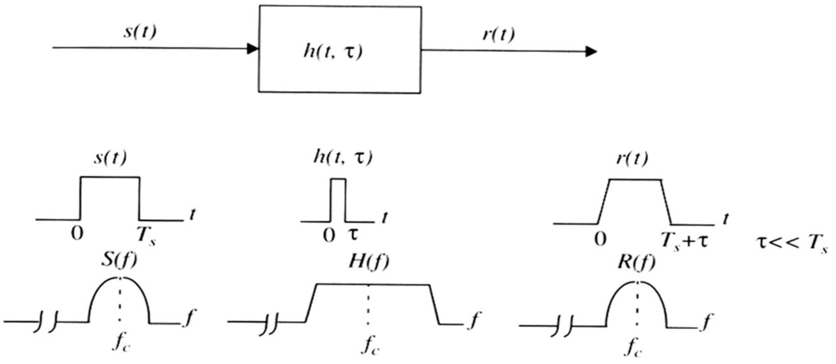


Figure 5.12 Flat fading channel characteristics.

Flat Fading

- Mobile radio channel has a const gain and linear phase response over a bw which is greater than bw of tx signal.
- The multipath structure of channel is such that spectral char of tx signal are preserved at rx.
- Strength of rx signal changes with time, due to fluctuations in gain of channel caused by multipath.
- Causes deep fades, may require 20 or 30 dB more tx power to achieve low ber during times of deep fades as compared to systems operating over nonfading channels.
- Also known as <u>amplitude varying channel</u>.
- Also referred to as <u>narrowband channels</u> since bw of applied signal is narrow as compared to channel flat fading bw.
- The most common amplitude distribution of flat fading channel is *Rayleigh distribution*.

Frequency Selective Fading

- Signal undergoes frequency selective fading if W is > $(\Delta f)_c$ & $T_s > T_m$

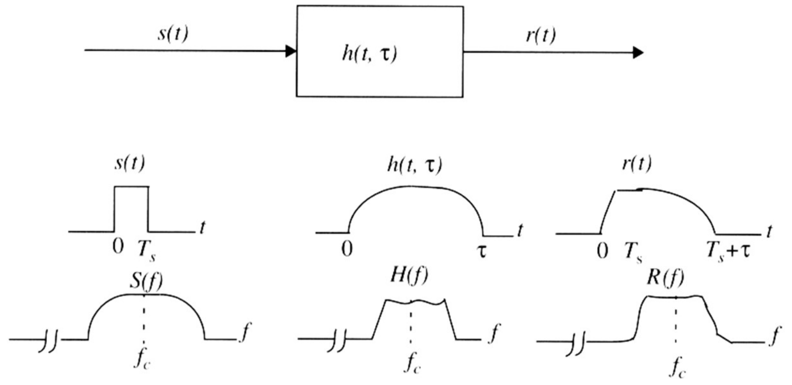


Figure 5.13 Frequency selective fading channel characteristics.

- Channel has a constant-gain and linear phase response over a bw that is smaller than the bw of tx signal.
- Received signal includes multiple versions of the tx waveform which are attenuated and delayed in time.
- Channel induces inter-symbol interference.
- Certain frequency components in the rx signal spectrum have greater gains than others.
- Also known as wideband channels.
- *M*ary Rayleigh fading model is usually used for analyzing frequency selective small-scale fading.

Types of Small-Scale Fading

- Based on Doppler Spread
  - Fast Fading
    - High Doppler spread.
    - Coherence time < Symbol Period.
    - Channel variations faster than base-band signal variations
  - Slow Fading
    - Low Doppler spread.
    - Coherence time > Symbol period.
    - Channel variations slower than base-band signal variations

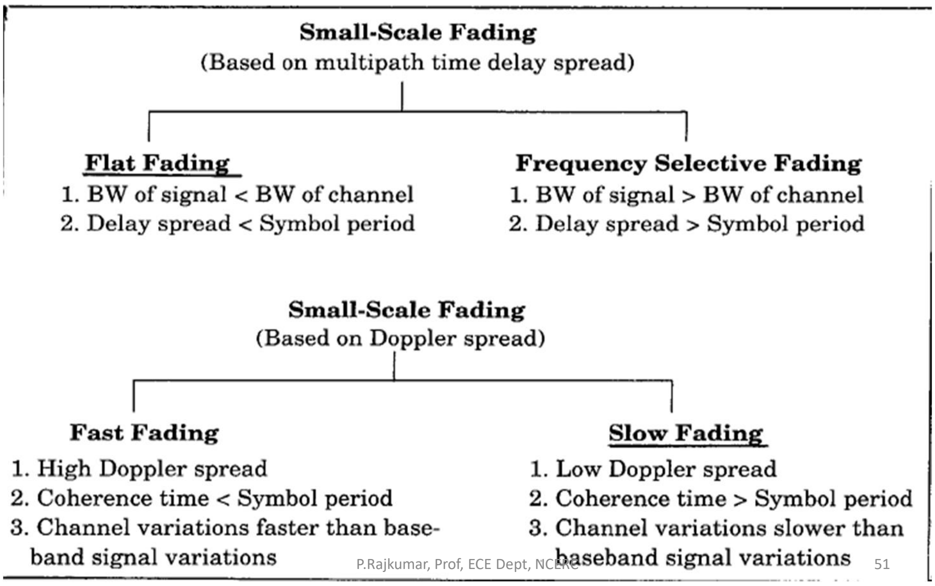
Fast Fading

- Channel impulse response changes rapidly within symbol duration.
- Coherence time of channel is smaller than symbol period of tx signal.
- Signal distortion due to fast fading increases with increasing Doppler spread relative to bw of tx signal.
- Fast fading only deals with rate of change of channel due to motion.
- In practice, fast fading only occurs for very low data rates (or very fast motion speed).

Slow Fading

- Channel impulse response changes at a rate much slower than tx baseband signal $s(t)$.
- Channel may be assumed to be static over one or several reciprocal bw intervals.
- Doppler spread of channel is much less than the bw of baseband signal.
- The velocity of mobile and baseband signaling determines whether a signal undergoes fast fading or slow fading.

Type of Small-Scale Fading



Small-Scale Fading
(Based on multipath time delay spread)

Flat Fading
1. BW of signal < BW of channel
2. Delay spread < Symbol period

Frequency Selective Fading
1. BW of signal > BW of channel
2. Delay spread > Symbol period

Small-Scale Fading
(Based on Doppler spread)

Fast Fading
1. High Doppler spread
2. Coherence time < Symbol period
3. Channel variations faster than base-band signal variations

Slow Fading
1. Low Doppler spread
2. Coherence time > Symbol period
3. Channel variations slower than baseband signal variations

FREQUENCY-NONSELECTIVE, SLOWLY FADING CHANNEL

- To derive <u>error rate performance</u> of binary modulated signals tx over a freq-nonselective, slowly fading channel.
- This results in ⊗ distortion of tx signal $s_l\,(t)$.
- The condition that channel fades slowly→ the ⊗ process may be regarded as a const during at least one signaling interval.
- the rx equi lp signal in one signaling interval is:

$$r_l(t) = \alpha e^{j\phi} s_l(t) + z(t), \qquad 0 \le t \le T \qquad \textbf{34}$$

- where α is <u>atten</u> and $z(t)$ represents the complex-valued <u>white Gaussian noise</u> random process corrupting the signal.

- Assume channel fading is sufficiently slow that phase shift ϕ can be estimated from rx signal without error.
- Ideal coherent detection of the rx signal.
- Rx signal can be processed by passing it thro' a matched filter in case of binary PSK or thro' a pair of matched filters in case of binary FSK.
- expression for the error rate of binary PSK as a function of the rx SNR

$$P_b(\gamma_b) = Q\left(\sqrt{2\gamma_b}\right) \quad 35$$

$$\text{where } \gamma_b = \alpha^2\mathcal{E}_b/N_0. \quad 36$$

- The expression for the error rate of binary FSK, detected coherently, is

$$P_b(\gamma_b) = Q\left(\sqrt{\gamma_b}\right) \quad 37$$

- To obtain error prob when **α** is random,

$$P_b = \int_0^\infty P_b(\gamma_b)p(\gamma_b)\,d\gamma_b \quad 38$$

- where $p(\gamma_b)$ is the **p**robability **d**ensity **f**unction of γ_b when **α** is random.
- γ_b is ratio of the signal energy per bit to noise power spectral density

PSK and FSK under Rayleigh fading

- When α is Rayleigh-distributed, α^2 has a Chi-square prob distrib with 2 deg of freedom.
- So γ_b also is chi-square distributed.

$$p(\gamma_b) = \frac{1}{\bar{\gamma}_b} e^{-\gamma_b/\bar{\gamma}_b}, \qquad \gamma_b \geq 0 \qquad 39$$

$$\overline{\gamma}_b = \frac{\mathcal{E}_b}{N_0} E(\alpha^2) \qquad 40$$

- subst eqn 39 into eqn 38, carry out integration for $P_b(\gamma_b)$ as given by eqns 35 and 36.

- The result of this integration for binary PSK is

$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}_b}{1+\bar{\gamma}_b}}\right) \qquad 41$$

- repeat integration with $Pb(\gamma b)$ given by eqn 36, prob of error for binary FSK, detected coherently,

$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}_b}{2+\bar{\gamma}_b}}\right) \qquad 42$$

DPSK under Rayleigh Fading

- If fading is quite rapid to prevent estimation of a stable phase ref by averaging rx signal phase over many signaling intervals, DPSK can be used as it requires phase stability over only two consecutive signaling intervals.
- This modulation technique is quite robust in the presence of signal fading.
- Error prob for a nonfading channel, is

$$P_b(\gamma_b) = \tfrac{1}{2}e^{-\gamma_b} \qquad \mathbf{43}$$

- Subs eqn 43 into integral in eqn 38 , use $p(\gamma_b)$ from eqn 39.
- Evaluation of resulting integral gives prob of error for binary DPSK,

$$P_b = \frac{1}{2(1+\overline{\gamma}_b)} \qquad \mathbf{44}$$

- A non-coherent (envelope or square-law) detector with binary, orthogonal FSK signals, has the error probability for a nonfading channel:

$$P_b(\gamma_b) = \tfrac{1}{2}e^{-\gamma_b/2} \qquad \mathbf{45}$$

- When $P_b(\gamma_b)$ is averaged over the Rayleigh fading channel attenuation, the resulting error probability is:

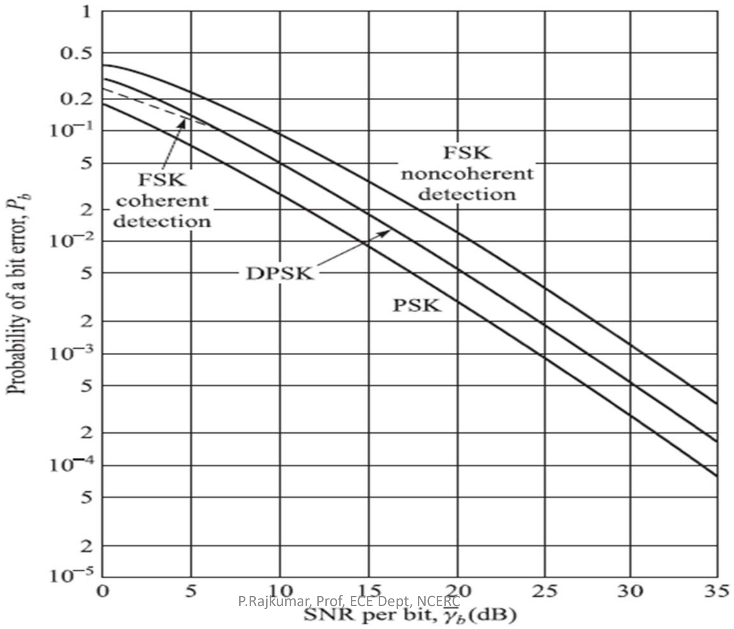
$$P_b = \frac{1}{2+\overline{\gamma}_b} \tag{46}$$

- In comparing the performance of four binary signaling systems, focus on the prob of error for large SNR, i.e., $\gamma_b >> 1$

$$P_b \approx \begin{cases} 1/4\overline{\gamma}_b & \text{for coherent PSK} \\ 1/2\overline{\gamma}_b & \text{for coherent, orthogonal FSK} \\ 1/2\overline{\gamma}_b & \text{for DPSK} \\ 1/\overline{\gamma}_b & \text{for noncoherent, orthogonal FSK} \end{cases} \tag{47}$$

Performance comparison of binary signaling on a Rayleigh fading channel.

- Coherent PSK is 3 dB better than DPSK and 6 dB better than non-coherent FSK.
- The error rates decrease only inv with SNR.
- In contrast, decrease in error rate on a nonfading channel is exp with SNR.
- So, on a fading channel, transmitter must transmit a large amount of power in order to obtain a low prob of error.
- In many cases, a large amount of power is not possible, technically and/or economically.


1
0.5
0.2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
Probability of a bit error, P_b
0
5
10
15
20
25
30
35
SNR per bit, $\overline{\gamma}_b$(dB)
FSK coherent detection
FSK noncoherent detection
DPSK
PSK

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Performance Comparison of Sampling Techniques

Parameter	Ideal Sampling	Natural sampling	Flat top sampling
Sampling Principle	Multiplication	Chopping	Sample & hold circuit
Feasibility	Non practical	Practical	Practical.
Sampling Rate	Infinity	$F_s \geq 2f_m$	$F_s \geq 2f_m$
Noise interference	Maximum	Minimum	Maximum

28|2|18

MODULE 6

Multipath channels.

Fading describes the rapid fluctuations of amplitudes of phases of a radio signal over a shot period of time or distance. Fading is caused by interference between 2 or more versions of the transmitted signal which arise at the receiver at different periods of times. These waves called multipath waves combine at the receiver to give a resultant signal which vary in amplitude & phase. Multipath effects creates small scale fading effects such as

a) rapid changes in signal strength

b) random frequency modulation

c) Time dispersion.

Coherence Bandwidth (Bc)

It is the measure of transmission bandwidth for which signal distortion across the channel becomes noticeable. It is the statistical measure of range of frequencies over which the channel can be considered flat. ie, the channel passes all components with equal gain and phase. If the frequency corelation is above 0.9 Coherence bandwidth is given by $1/50\sigma_\tau$ where σ_τ is the delay. If the frequency corelation is above 0.5 then the coherence BW is given by

$$B_c = \frac{1}{5\sigma_\tau}$$

Coherence Time (Tc)

It is the measure of transmitted signal duration for which the distortion across the channel becomes noticable. It is the time duration over which two received signals have strong amplitude corelation.

Coherence time, $(T_c) = \frac{9}{16\pi f_m}$

where f_m - maximum frequency.

Classification of Multipath channels

1) **Flat Fading Channel:**

If the channel has constant gain and linear phase response over a bandwidth which is greater than Bw of the transmitted signal the received signal will undergo flat fading. Flat fading channels are also known as amplitude varying channels or narrow band channels. For a flat fading channel. $B_s < B_c$ and $\sigma_\tau < T_s$.

B_s – Bandwidth of the signal

B_c – Bandwidth of the channel

σ_τ – delay

T_s – Symbol period.

2) **Frequency Selective Fading Channels:**

If the channel has a constant gain and linear phase over a bandwidth which is smaller than bandwidth of the transmitted signal. The channel creates frequency selective fading. This channel induces ISI. These channels are also known as wideband channels. For a frequency selective channel $B_s > B_c$, $\sigma_\tau > T_s$

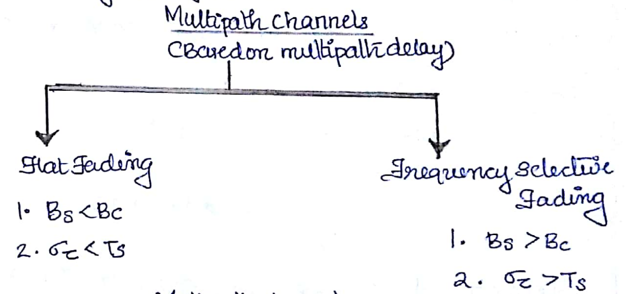
3) **Fast Fading Channels:**

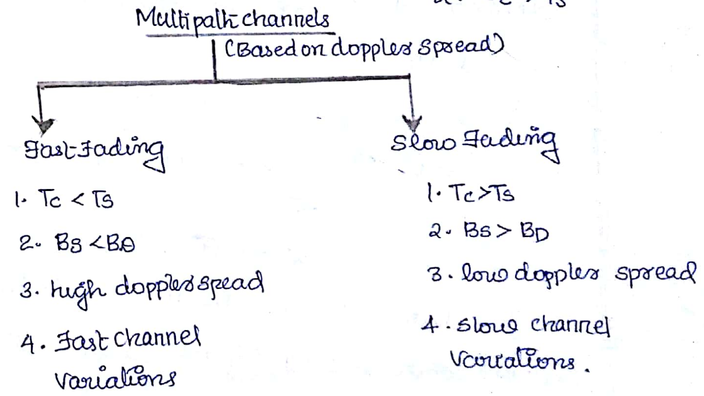
The channel impulse response changes rapidly with in the symbol duration. The coherence time of the channel is smaller than symbol period. The bandwidth of the transmitted signal is less than doppler bandwidth. This causes frequency dispersion leading to signal distortion. In fast feeding channel $T_c < T_s$ and $B_s < B_D$.

In flat fading, fast fading channel amplitude of the sgnl changes according to the variations in channel. In frequency selective fast fading channels amplitude, phase and time delay of the transmitted signal vary according to the channel variations

4) **Slow Fading channels:**

The channel impulse response changes at a rate slower than transmitted signal. The Bw of the transmitted signal is greater than doppler bandwidth. The coherence time is greater than symbol period.





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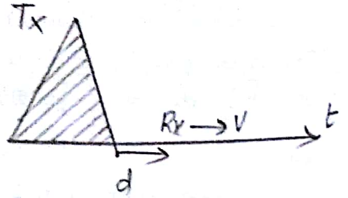
Friday

Statistical characterisation of Multipath channels / Impulse response of multipath channels

The small scale variations of a mobile radio signal is related to impulse response of the channel. The impulse response is a wideband characterisation that contains all information required to stimulate and analyse the transmission through channel



Assume a mobile receiver moving along the ground with a constant velocity 'v'. Let the distance be 'd'. The channel impulse response is ~~h(t)~~ h(d,t). with i/p x(t) and o/p Y(d,t). The o/p signal can be represented in terms of convolution as

$$Y(d,t) = x(t) \circledast h(d,t)$$

$$= \int_{-\alpha}^{\alpha} x(\tau)\, h(d, t-\tau)\, d\tau$$

distance d can be represented in terms of Velocity as

$d = vt$

∴ the o/p signal is given by

$$Y(Vt, t) = \int_{-\alpha}^{\alpha} x(\tau)\, h(Vt - t - \tau)\, d\tau$$

$$= x(\tau) \circledast h(Vt, t)$$

The channel can be modelled as linear time varying channel where the o/p signal varies with velocity and time.

Since, the received signal in a multipath channel has a series of attenuated time shifted and phase shifted versions of i/p signal the baseband impulse response of the channel is given by

$$h_b(t,\tau) = \sum_{i=0}^{N-1} a_i(t,\tau) \exp\left[j\{2\pi f_c \tau_i(t) + \phi_i(t,\tau)\}\right] \delta\left[\tau - \tau_i(t)\right]$$

where, $a_i(t,\tau)$ is the amplitude of the signal

$\tau_i(t)$ is the delay.

$2\pi f_c \tau_i(t) + \phi_i(t,\tau)$ is the phase shift.

Transmission over a Rayleigh Fading Channel

Rayleigh distribution is used to describe the time varying nature of the received signal. This model assumes that the magnitude of a signal that has travelled through a comm. channel will vary randomly according to the Rayleigh distribution. It is a reasonable model when there are many objects in the propagation path of the signal that scatter the radio signal before it arise at the receiver. The probability density function of Rayleigh distribution is given by

$$P_R(\gamma) = \begin{cases} \frac{\gamma}{\sigma^2} \exp\left(\frac{-\gamma^2}{2\sigma^2}\right) & 0 \le \gamma \le \infty \\ 0 & \gamma < 0 . \end{cases}$$

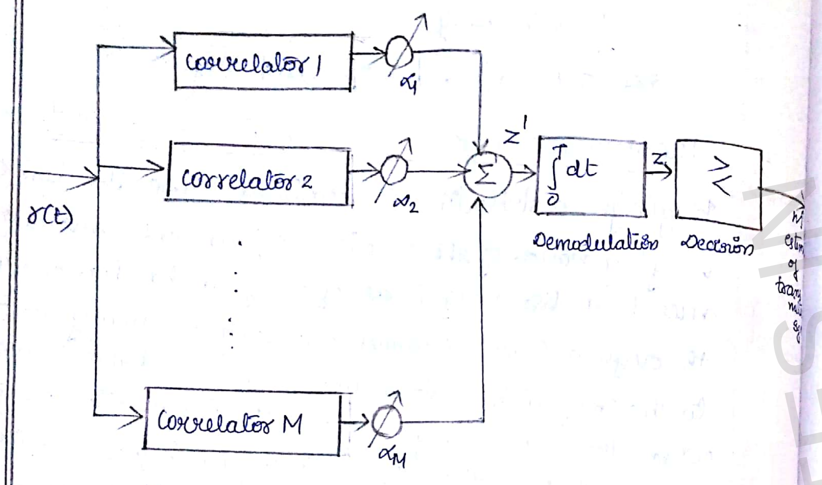
where σ - RMS value of received signal

γ - Rayleigh factor

P_R - Probability of received sgnl.

Rake Receiver.

It is a radio receiver designed to overcome the effects of multipath fading. It uses several corelators assigned to each multipath component. Each corelator independently decodes single multipath component. The o/p's of each corelator are weighted to provide better estimate of transmitted sgnl. Demodulation & decision are based on the o/p of the correlators. The rake receiver improves signal to noise ratio.



$$z' = \sum_{m=1}^{M} \alpha_m z_m \qquad \alpha_m = \frac{z_m^2}{\sum_{m=1}^{M} z_m^2}$$

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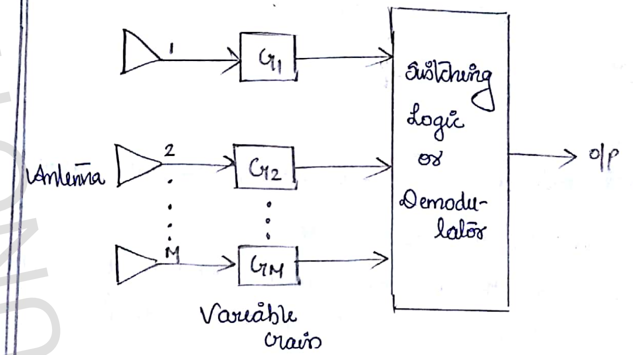
Diversity & Diversity Techniques

Diversity is a powerful tool in communication that provides wireless link improvement at a low cost. It is a method for improving the reliability of msg sgnl by using two or more communication channels with diff characteristics. If one of the comm. channels undergo a deep fade the other channels can compensate the effect by strongest signal. This method is implemented by using 2 or more receiving antennas. It reduces the depth and duration of fades experienced by the receiver. This method is used to develop information from several signals transmitted over independent fading paths. Diversity improves the signal to noise ratio at the receiver.

Types of Diversity

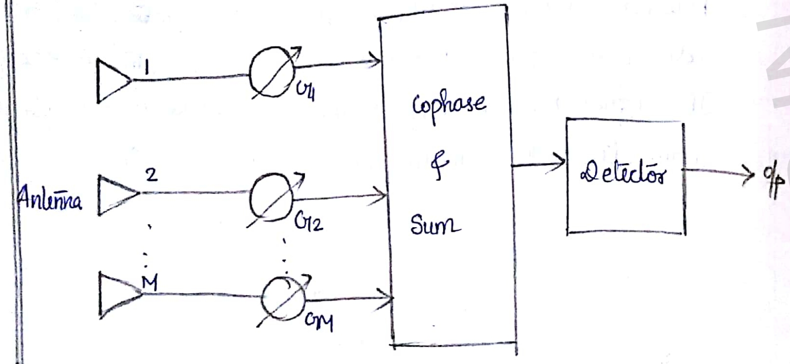
1. Space Diversity : It is also known as antenna diversity. Multiple receiving antennas are used to provide diversity at the receiver. This method is classified as

a) Selection Diversity - It is the simplest diversity technique. M branches provide multipath signals whose gains are adjusted to provide avg signal to noise ratio for each branch. The signal branch having highest signal to noise ratio is connected to the demodulator to produce the o/p



b) Feedback or Scanning Diversity - Instead of always using the best of M signals the M signals are scanned in a fixed sequence until one is found to be above a predetermined threshold. This sgnl is received until it falls below the threshold and the scanning process is initiated again.

c) Maximal Ratio Combining (MRC) - The sgnls from all of the ~~m~~ M branches are ~~wa~~ weighted according to their individual signal voltage and power and are added. Individual signals must be cophased before adding. This requires cophase and summing element. This method produces a signal to noise ratio equal to individual signal to noise ratios. This method has the advantage of producing an o/p with an acceptable sgnl to noise ratio E1 when sum of the individual signals are not acceptable



d) Equal Gain Combining – In certain cases it is not convinient to provide variable weighting gains as in MRC. In such cases the branch weights are set to unity. Signals from each branch are cophased and added.

2. Polarisation Diversity – Multiple Versions of a transmitted signal are passed through antenna's with different polarisation. Circular, linear, vertical and horizontal polarised antennas are used to reduce multipath effects

3. Frequency Diversity. – The signal is transmitted using several frequency channels that are effected by frequency selective fading. Frequencies seperated by more than coherence bandwidth of the channel will not experience fading.

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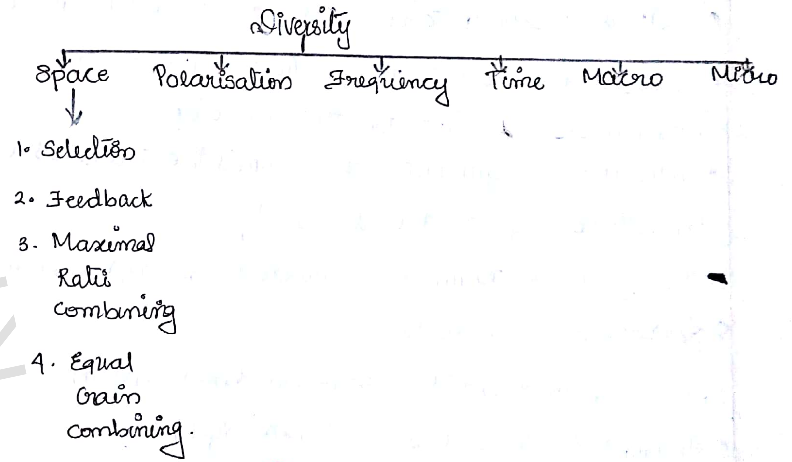
4. Time Diversity :– Multiple Versions of the same sgnl are transmitted at diff time instance. The information is transmitted at time spacings that exceed the coherence time of the channel. Time diversity uses rake receiver.

5) Macroscopic Diversity – This method prevents large scale fading. Large scale fading is caused by shadowing due to nature of the surroundings.

6. Microscopic Diversity: It prevents small scale fading. Small scale fading is caused by multiple reflections from the surrounding. It causes rapid amplitude fluctuations.



Multiple Access Techniques

Multiple Access Schemes are used to allow many users to share simultaneously a finite amount of radio spectrum. The sharing of spectrum increases the capacity. Frequency division multiple access (FDMA), time division multiple access (TDMA) and code division multiple access (CDMA) are the 3 major techniques used to share the available bandwidth in a communication system. The multiple acc techniques are also divided as narrow band systems and wide band systems. In narrow band systems the transmission BW of the channel is lower than coherence bandwidth. In wide band systems the transmissi BW of the channel is larger than coherence BW.

Frequency Division Multiple Access (FDMA)

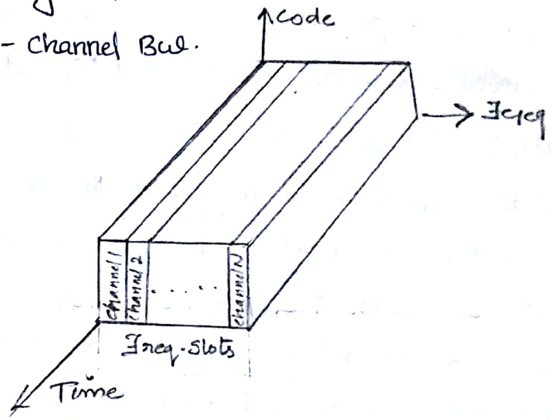
- FDMA assigns individual channels to individual users.
- During the period of data transmission no other user can share the same channel.
- FDMA channel carries only one data stream at a time.
- If a FDMA channel is not in use it cannot be used by other users to increase the capacity.
- After the assignment of channels the antenna's transmit simultaneously and continuously.
- BW of FDMA channels are narrow. (30kHz) Each channel
- So supports only one user.
- FDMA is implemented as narrow band system
- Amount of ISI is low in FDMA systems.
- Complexity of FDMA is low.
- FDMA systems are costly.
- FDMA systems require filtering to minimize interference.
- No: of channels supported by a FDMA system is given by

$$N = \cancel{Bt}\ \frac{B_t - 2B_g}{B_c}$$

B_t - total BW.

B_g - guard BW

B_C - channel BW.



Time Division Multiple Access (TDMA)

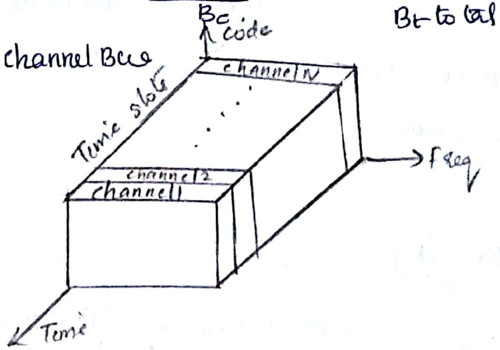
- TDMA system divide the radio spectrum into time slot.
- In each time slot one user is allowed to transmit or receive data.
- TDMA shares a single carrier frequency with several users where each user makes use of non overlapping time slots.
- This method has low battery consumption.
- It uses diff. time slots for transmission & reception.
- High transmission rate.
- high synchronisation is required.
- It is possible to allocate diff no: of timeslot to diff users.
- The total no: of time slots in TDMA is given by

$$N = \frac{m(B_t - 2B_g)}{B_c}$$

m - max no: of users.

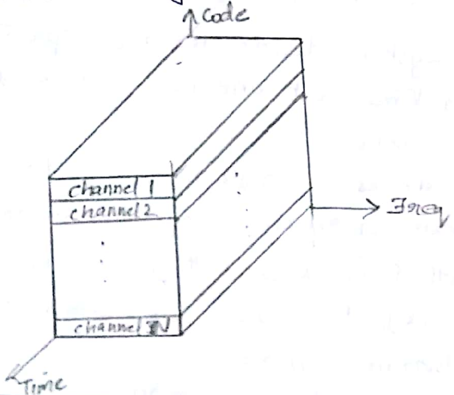
B_t - total BW, B_g - guard BW

B_c - channel BW



Code Division Multiple Access (CDMA)

- In CDMA system narrowband msg sgnl is multiplied by a very large BW sgnl called 'Spreading sgnl'.
- Each user has its own code word.
- For the detection of msg sgnl the receiver needs to know the code word used by the transmitter.
- Users have same freq & time period.
- Channel data rates are very high in CDMA.
- Each channel is assigned a code word which is orthogonal to the code words used by other users.



Problem

1. In a FDMA system 416 channels are allowcated to users. The channel BW is 30kHz and the guard BW is 10kHz. Calculate the total spectrum allowcated.

Sol no: of channel, N = 416

$B_c = 30kHz$

$B_g = 10kHz$

$$N = \frac{B_t - 2B_g}{B_c}$$

$$B_t = NB_c + 2B_g = (416)(30 \times 10^3) + 2(10 \times 10^3)$$
$$= 12.5MHz$$

2. Consider a FDMA system that uses 25MHz BW broken down into channels of BW 200kHz each. If there is no guardband. Find the total no: of time slots for a max of 8 users.

Sol. $B_c = 200kHz$, $B_t = 25MHz$, $B_g = 0$, $m = 8$.

$$N = \frac{m(B_t - 2B_g)}{B_c}$$

$$= \frac{8(25 \times 10^6)}{200 \times 10^3}$$

$$= \underline{1000} \text{ time slots.}$$

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Orthogonal Frequency Division Multiplexing (OFDM)

OFDM technology splits high rate data stream into a number of low rate datastreams that are transmitted simultaneously through sub carriers. It combines modulation and multiplexing. The symbol duration is increased for low rate sub carriers there by reducing multipath delay spread. In OFDM systems the spectrum of individual sub carrier is overlapped with minimum frequency spacing. It is a special case of multicarrier transmission. Multicarrier transmission refers to the transmission technique in which the user can employ large no: of carriers to transmit data simultaneously. In OFDM the carrier signals are orthogonal to one another. The sub carrier spacing is given by

$$\Delta F = n/T_u$$

where T_u is the symbol duration

n - +ve integer.

The total BW of OFDM is given by

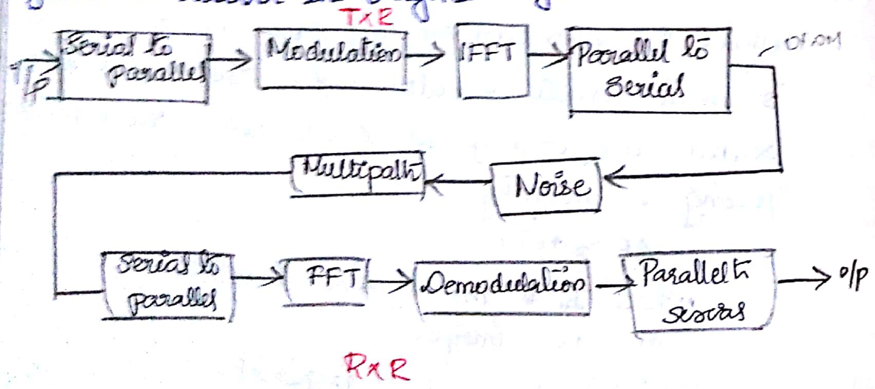
$$B = \cancel{N} \Delta F . N$$

N - no: of users

OFDM uses guard interval between the symbols to eliminate ISI. The guard interval also eliminates the need for pulse shaping filter there by reducing time delay problems.

The transmitter of OFDM converts the i/p data from serial to parallel form. Each set of data contains one information bit for each carrier frequency. The parallel data are then modulated. The IFFT converts the parallel data into time domain waveforms. The parallel to serial converter creates the OFDM signal by sequentially outputing the time domain samples.

The receiver performs the inverse of transmitter by 1st converting the data into parallel form. The FFT converts these parallel data streams into frequency domain data. This data is demodulated. Demodulation is basically down conversion of the information back to original signal. Finally this parallel data is converted back into serial form to recover the original signal.



Advantages

1. Resistant to fading
2. No data is lost due to ISI.
3. highly efficient.
4. low rate of transmission.
5. low BW requirements.
6. Simple channel equalisation.

Disadvantages

1. high pwr requirements
2. Sensitive to freq shift.